Chapter 16. Gravity Waves

- *What are gravity waves?*
- *How do gravity waves differ from ocean wave? from electromagnetic waves?*
- *How does the detection of gravity waves differ from detection of light waves?*
- *I hear that the current gravity wave detector is called LIGO. What does the name stand for?*
- *How does LIGO detect gravity waves?*
- *Even without LIGO, is there evidence for gravity waves?*

CHAPTER 16

Gravity Waves

Edmund Bertschinger & Edwin F. Taylor *

Primordial gravity waves would be fossils from the very instant of creation . . . No other signal survives from that era.

—Marcia Bartusiak

16.1 ■ GENERAL RELATIVITY PREDICTS GRAVITY WAVES

Gravity wave: a tidal acceleration that propagates through spacetime.

General relativity differs from Newtonian gravity in several important ways. One way is in worldlines of stones and light in strong gravitational fields, especially near a black hole. The black hole was predicted by Michell and Laplace on the basis of Newtonian gravity more than a century before Schwarzschild derived his famous metric. However, the event horizon, singularity, and no-hair theorems are all consequences of general relativity that could not have been predicted from Newtonian physics.

Newton: Gravity propagates instantaneously.

Gravitational radiation is another phenomenon predicted by general relativity that has no counterpart in Newtonian physics. Without quite saying so, Newton assumes that the gravitational interaction propagates instantaneously: When the Earth moves around the Sun, the Earth's gravitational field changes all at once everywhere.

Einstein: No signal propagates faster than light.

When Einstein formulated special relativity and recognized its requirement that no information can travel faster than the speed of light in a vacuum, he realized that Newtonian gravity would have to be modified. Not only would static gravitational fields differ from the Newtonian prediction in the vicinity of compact masses, but also time-varying fields would propagate. He showed that these fields would move with the speed of light, so even gravity cannot be used to send information faster than the speed of light.

Are gravity waves like electromagnetic waves?

Einstein had a conceptual prototype for gravity waves: electromagnetic radiation. James Clerk Maxwell predicted electromagnetic radiation in 1873, Einstein was born in 1879, and Heinrich Hertz demonstrated electromagnetic waves experimentally in 1888. When he grew up, Einstein quickly realized that

*Draft of Second Edition of *Exploring Black Holes: Introduction to General Relativity*

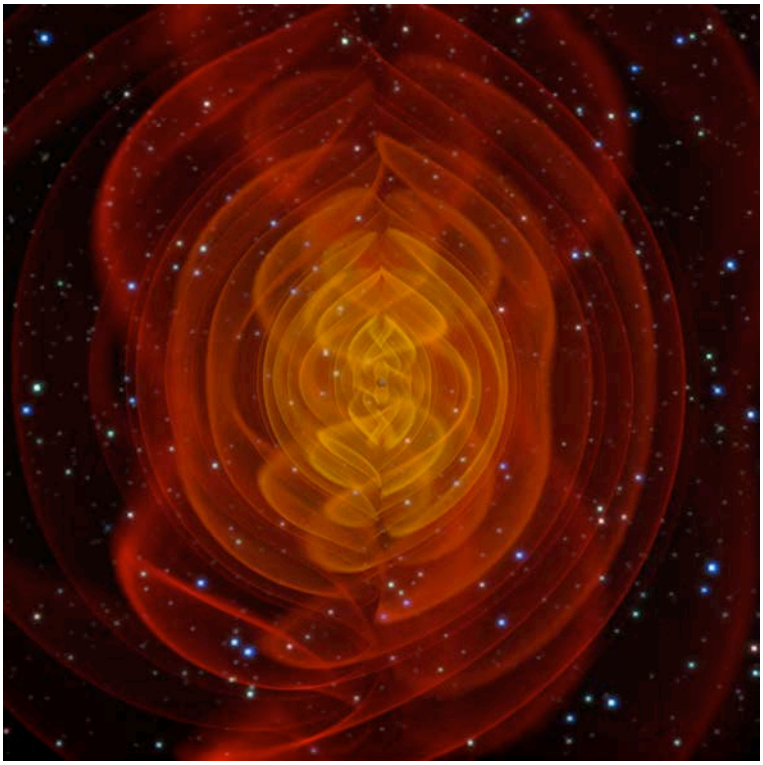

FIGURE 1 Computed emission of gravity waves. The tiny dot at the center of this image is two black holes churning spacetime as they combine into one. The swirling patterns represent distortions of spacetime that propagate outward as gravity waves. Close to the coalescing black holes, the gravity waves—essentially nothing but traveling tidal accelerations—are lethal. We hope to detect extremely small-amplitude gravity waves on Earth.
(#BHmerge)

a general relativity theory would not look like Maxwell's electromagnetic theory. When general relativity theory was completed, Einstein and others were able to compute how gravitational fields propagate. Gravity waves of small amplitude propagate with the speed of light.

Gravity wave: propagating tidal accelerations

What do we mean by "gravity waves"? Gravity waves are tidal accelerations that vary with time and position; that is all they are. As a gravity wave passes over you, you are alternately stretched and compressed in ways that depend on the particular form of the wave. In principle there is no limit to the amplitude of a gravity wave. Figure 1 pictures the calculated result of two black holes emitting gravity waves as they combine into one. In the vicinity of the coalescence, gravity-wave-induced tidal forces would be dangerous to life.

Gravity waves from various sources continually sweep over us on Earth. Sections 16.3 and 16.7 describe some of these sources. Basically we hope to observe these waves by detecting changes in separation between two test masses suspended near to one another—changes in separation caused by the

Gravity wave on Earth: An extremely small traveling tidal effect

traveling tidal effects that constitute a gravity wave. We expect this change in separation to be *extremely* small for gravity waves likely to be detected on Earth.

Gravity wave detectors are interferometers.

Current gravity wave detectors on Earth are interferometers in which light is reflected back and forth between "free" test masses positioned at the ends of two perpendicular vacuum chambers, and the time difference measured between round-trip times in the two directions. The "free" test masses are hung from wires that are in turn supported on elaborate shock-absorbers to minimize the vibrations from passing trucks and even waves crashing on a distant shore. But the pendulum-like motions of these test masses are free enough to permit measurement of their change in separation due to tidal effects of a passing gravity wave, caused by some gigantic distant event such as the coalescence of two black holes modeled in Figure 1.

Gravity waves not disturb interferometer structures

Question: Does the change in separation induced by gravity waves affect everything, for example a meter stick or the concrete slab on which a gravity wave detector rests? *Answer:* Only by an amount that is entirely negligible. The structure of meter sticks and concrete slabs is determined by electromagnetic forces mediated by quantum mechanics. The two ends of a meter stick are not freely-floating test masses. The tidal force of a passing gravity wave is much weaker than the internal forces that maintain the shape of a meter stick—or the concrete slab supporting the vacuum chamber of a gravitational-wave observatory; these are stiff enough to be negligibly affected by a passing gravity wave.

Comment 1. "Gravity waves" in deep water

Spacetime waves are often called **gravitational waves** to distinguish them from *gravity waves*. Classical gravity waves in water are deep-water waves, whose form is not influenced by the bottom or other boundary conditions. In this book the term *gravity waves* refers exclusively to propagation of tidal effects in spacetime.

16.2 ■ GRAVITY WAVE METRIC

Tiny but significant departure from the inertial metric

Gravity wave metric

Our analysis uses a particular gravity wave: a plane wave from a distant source that moves in the z-direction. Every gravity wave we discuss in this chapter (except those shown in Figure 1) represents a very small deviation from flat spacetime. Here is the metric for a gravitational plane wave that propagates along the z-axis. (#GravWaveMetric)

$$d\tau^2 = dt^2 - (1+h)dx^2 - (1-h)dy^2 - dz^2 \qquad (h \ll 1) \tag{1}$$

$h =$ gravity wave strain

In this metric h is the tiny fractional deviation from the flat-spacetime coefficient of dx^2 and dy^2 in an inertial metric. A technical name for fractional deviation of length is **strain**, so h is also called the **gravity wave strain**. Metric (1) describes a transverse wave, since h describes a perturbation of

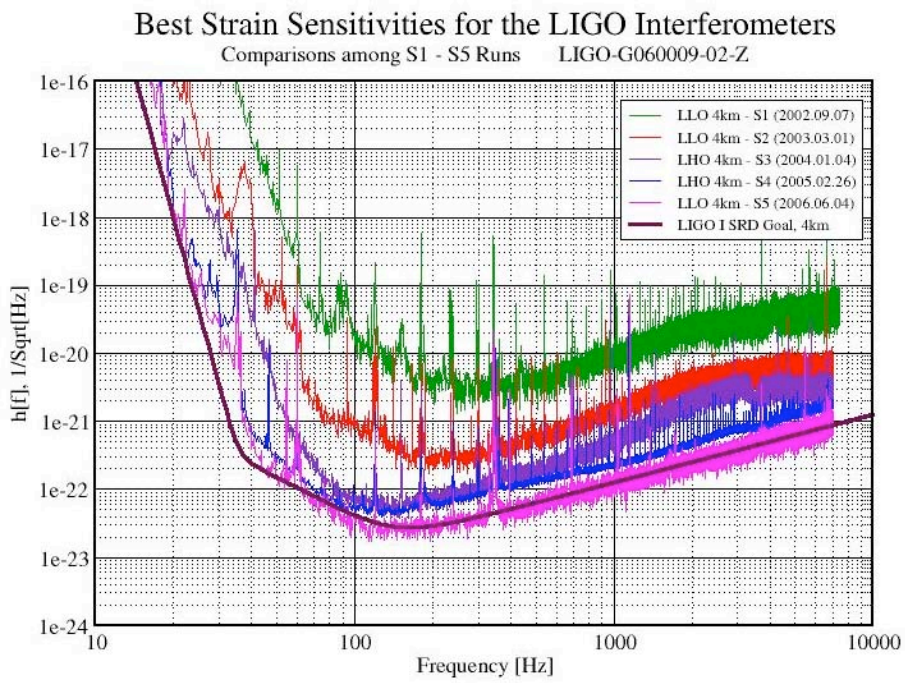

FIGURE 2 Progressive improvements in sensitivity of LIGO interferometers. On the vertical axis $h =$ 1e-19, for example, means a fractional change in separation of 10^{-19} between test masses. The bottom solid line is the current goal. Spikes occur at frequencies of electrical or acoustical noise. To be detectable, gravity wave signals must cause greater fractional change than what is represented by these noise curves. In 2015, gravity wave signals have yet to be detected.

(#LIGOsensitivity)(#NEED NEW PLOT)

space in the x and y directions transverse to the z-direction of propagation. We shall see that strain h varies with both position and time.

Let two free test masses be at rest a distance D apart in the x or y direction. When a z-directed gravity wave passes over them, the change in their separation, called the **displacement**, equals $h \times D$, which follows directly from the definition of h as a "fractional deviation."

One can use Einstein's field equations to make predictions about the magnitude of the function h in equation (1) for various kinds of astronomical phenomena. Currently, gravity wave detectors use laser interferometry and go by the full name **Laser Interferometer Gravitational Wave Observatory**, or **LIGO** for short. The first-generation LIGO, called *Initial LIGO*, completed in 19XX, was able to detect waves with (approximately) $h > 10^{-19}$ for frequencies within a range of about 100 hertz. (Recall that one hertz—abbreviation: Hz—is one cycle per second.) The second-generation LIGO, called Advanced LIGO, completed in 20XX, is about 10 times more

LIGO gravity wave detector

sensitive. Advanced LIGO can be tuned in frequency to achieve higher sensitivity in frequency bands of interest.

LIGO sensitivity

Figure 2 compares the gradually-improving sensitivities of LIGO over time. NEEDS LATEST VERSION. The displacement sensitivity is expressed in the units of meter/(hertz)$^{1/2}$ because the amount of noise limiting the measurement grows with the frequency range being sampled. Note that the instruments are designed to be most sensitive near 150 hertz. This frequency is determined by the different kinds of noise faced by experimenters: Quantum noise limits the sensitivity at high frequencies, while seismic noise is the largest problem at low frequencies. If the range of sampled frequencies—*bandwidth*—is 100 hertz, then the best current sensitivity is about $10^{-22} \times 100^{1/2} = 10^{-21}$. This means that along a length of 4 kilometers $= 4 \times 10^3$ meters, the change in length is approximately $10^{-21} \times 4 \times 10^3 = 4 \times 10^{-18}$ meters, which is approximately one thousandth the size of a proton, or a hundred million times smaller than a single atom!

?

Objection 1. *Your gravity wave detector sits on Earth's surface, but equation (1) says nothing about curved spacetime described, for example, by the Schwarzschild metric. The expression $2M/r$ measures departure from flatness in the Schwarzschild metric. At Earth's surface, $2M/r \approx 1.4 \times 10^{-9}$, which is 10^{13}—ten million million!—times greater than the corresponding gravity wave factor $h \sim 10^{-22}$. Why doesn't the quantity $2M/r$—which is much larger than h—appear in (1)?*

!

The factor $2M/r$ is essentially constant across the structure of LIGO, so we can ignore its change as the gravity wave sweeps over it. Indeed, the LIGO detector is "tuned" to detect a time-varying gravity wave of frequency near 150 hertz. LIGO is totally insensitive to the *static* curvature introduced by the factor $2M/r$ at Earth's surface. For this reason, we simply omit static curvature factors from equation (1), effectively describing gravity waves "in free space" for the predicted $h \ll 1$.

Einstein's equations become a wave equation.

In free space and for small values of h, Einstein's field equations actually reduce to a wave equation for h. For the most general case, this wave has the form $h = h(t, x, y, z)$. When t, x, y, z are all expressed in meters, this wave equation takes the form: (#GravWaveEqn)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 h}{\partial t^2} \qquad \text{(free space and } h \ll 1) \tag{2}$$

For simplicity, think of a plane wave moving along the z-axis. The most general solution to the wave equation under these circumstances is (#zsolution)

$$h = h_{+z}(z - t) + h_{-z}(z + t) \tag{3}$$

Assume gravity wave moves in $+z$ direction.

The expression $h_{+z}(z - t)$ means a function h of the single variable $z - t$. The function $h_{+z}(z - t)$ describes a wave moving in the positive z-direction

and the function $h_{-z}(z+t)$ describes a wave moving in the negative z-direction. In this chapter we deal only with a gravity wave propagating in the positive z-direction (Figure 5) and hereafter set (#Simplifiedh)

$$h \equiv h(z-t) \equiv h_{+z}(z-t) \qquad \text{(wave moves in } +z \text{ direction)} \tag{4}$$

The argument $z-t$ means that h is a function of *only* the combined variable $z-t$. Indeed, h can be *any function whatsoever* of the variable $(z-t)$. The form of this variable tells us that, whatever the profile of the gravity wave at any given time (in the local detector frame), that profile displaces itself in the positive z-direction with the speed of light (local light speed = one, in our units) .

LIGO sensitive 75 to 500 hertz

Figure 2 shows that the LIGO gravity wave detector has maximum sensitivity for frequencies between 75 and 500 hertz, with a peak sensitivity at around 150 hertz. Even at 500 hertz, the wavelength of the gravity wave is very much longer than the overall 4-kilometer dimensions of the LIGO detector. Therefore *we can assume in the following that at any given time (in the local detector frame) the value of h is spatially uniform over the entire LIGO detector.*

QUERY 1. Uniform h?
Using numerical values, verify the claim in the preceding paragraph that h is effectively uniform over the LIGO detector.

Analogy: draw global map coordinates on rubber sheet.

It is important to understand that coordinates in metric (1) are global and to recall that global coordinates are arbitrary; we choose them to reveal aspects of a spacetime we cannot visualize. For $h \neq 0$, these global coordinates are invariably distorted. Think of the three mutually perpendicular planes formed by pairs of space coordinates (x, y), (y, z), and (z, x). Draw a grid of lines on a rubber sheet lying in each corresponding plane. By analogy, the passing gravity wave distorts these rubber sheets.

Gravity wave distorts rubber sheet.

Glue map clocks to the intersections of these grid lines on a rubber sheet so that they move as the rubber sheet distorts. A gravitational wave moving in the $+z$ direction (Figure 3) passes through a rubber sheet and acts in different directions within the plane of the sheet (Figures 3 and 4). The map clocks glued at intersections of map coordinate grid lines ride along with the grid as the sheet distorts, so the map coordinates of any clock do not change.

Map time t read on clocks glued to the rubber sheet.

Think of two ticks on a single map clock. Between ticks the map coordinates of the clock do not change: $dx = dy = dz = 0$. Therefore metric (1) tells us that the wristwatch time $d\tau$ between two ticks is also map time dt between ticks. Map time t corresponds to the time measured on the clocks glued to the rubber sheet, even when the strain h varies at their locations.

Figure 3 represents the map t-variation of the space distortion of the rubber sheet at a given location due to a particular polarization of the gravity

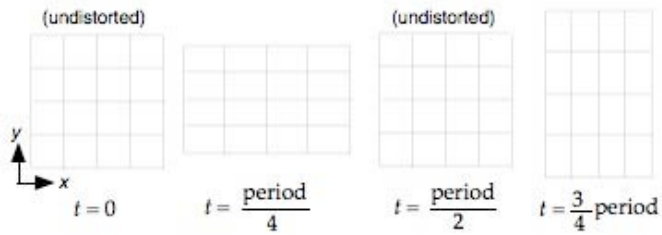

FIGURE 3 Change in shape (greatly exaggerated!) of the map coordinate grid at the same x, y location at four sequential times as a periodic gravity wave passes through in the z-direction (perpendicular to the page). NOTE carefully: The x-axis is stretched while the y-axis is compressed and vice versa. The areas of the panels remain the same.
(#GravWaveDistortion)

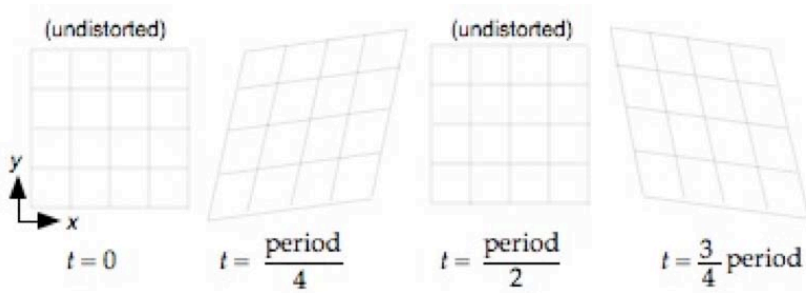

FIGURE 4 Effects of a periodic gravity wave with polarization "orthogonal" to that of Figure 3 on the map grid in the xy plane. Note that the axes of compression and expansion are at 45 degrees from the x and y axes. All grids stay in the xy plane as they distort. As in Figure 3, the areas of the panels are all the same.
(#GravWaveDistortionOrthog)

wave. Although gravity waves are transverse like electromagnetic waves, the polarization forms of gravity waves are different from those of electromagnetic waves. Figure 4 shows the distortion caused by a polarization "orthogonal" to that shown in Figure 3.

16.3 ■ SOURCES OF GRAVITY WAVES

Many sources; only one type leads to a clear prediction

Sources of gravity waves include collapsing stars, exploding stars, stars in orbit around one another, and the Big Bang itself. Neither electromagnetic waves nor gravity waves result from a spherically symmetric distribution of charge (for electromagnetic waves) or matter (for gravitational waves), even when that spherical distribution pulses symmetrically in and out (Birkhoff's Theorem, Section 6.5). Therefore, *symmetric* collapses and explosions emit no

No linear "antenna" for gravity waves

waves, either electromagnetic or gravitational. The most efficient source of electromagnetic radiation, for example along an antenna, is oscillating pairs of electric charges of opposite sign moving back and forth along the antenna, the resulting waves technically called **dipole radiation**. But mass has only one "polarity" (there is no negative mass), so there is no gravity dipole radiation from masses that oscillate back and forth along a line. Emission of gravity waves requires *asymmetric* movement or oscillation; the technical name for the simplest result is **quadrupole radiation**. Happily, most collapses and explosions are asymmetric; even the motion in a binary system is sufficiently asymmetric to emit gravitational waves.

We study here gravity waves emitted by a binary system consisting of two neutron stars—or a neutron star and a black hole—orbiting about one another (Section 16.7). All such pairs that we have detected are too far away to see directly. STILL TRUE? If we are to detect this binary system electromagnetically, at least one neutron star needs to be a **pulsar** that emits a steady stream of pulses that we can receive at a great distance. Pulsars turn out to be extremely stable clocks. As the two objects orbit, they also emit gravity waves that cause the binary system to lose energy, so that the orbiting objects gradually spiral in toward one another. These orbits are well described by Newtonian mechanics until about one millisecond before the two objects coalesce.

Binary system emits gravity waves . . .

Emitted gravity waves are nearly periodic during the Newtonian phase of orbital motion. As a result, these particular gravity waves are easy to predict and hence to search for. When the two objects coalesce, they emit a burst of gravity waves (Figure 11). After coalescence the resulting structure vibrates ("rings down"), emitting more gravity waves as it settles into its final state as a black hole. Initial LIGO has already completed its efforts and would have been sensitive enough (Figure 2) to detect binary neutron star systems coalescing at a distance of about 26 million light years. Unfortunately, no such coalescences were detected during more than one year of observation. Advanced LIGO extends the detection radius to 200 Megaparsecs $\approx$ 650 million light years. The volume of space increases approximately as the cube of the distance, so the improved sensitivity will vastly increase the number of galaxies that can be "seen" by LIGO from about one thousand to millions, increasing the odds of success thousands of times.

. . . whose amplitude is predictable.

Comment 2. Amplitude, not squared amplitude
The detection of a gravitational wave measures the *amplitude* or *strain* h of the wave. Received amplitude from a small source decreases as the inverse *distance.* In contrast, our eyes and other detectors of light respond to its *intensity,* which is proportional to the square of its amplitude, so detection of the intensity of light decreases as the inverse *square* of the distance.

QUERY 2. Increased volume of detection
Use numerical values given in the preceding paragraph to calculate to two significant figures the increased "odds of success" of Advanced LIGO compared with Initial LIGO.

From other sources: hard to predict.

Binary coalescence is the only source for which we can currently make a clear prediction of the signal (and therefore of the detection distance limit). Other conceivable sources include supernovae and the collapse of a massive star to form a black hole—the event that triggers a so-called **gamma-ray burst**. But we have only speculations about how far away any of these can be and still be detectable by either Initial LIGO or Advanced LIGO.

Comment 3. Detectors do not affect gravity waves
We are used to the fact that metal structures can distort or reduce the amplitude of electromagnetic waves passing across them. Even the presence of a receiving antenna can distort an electromagnetic wave in its vicinity. The same is not true of gravity waves, whose generation or modification requires massive moving structures. Gravity wave detectors have negligible effect on the waves that they are designed to detect.

QUERY 3. Electromagnetic waves *vs.* gravity waves. Discussion.
What property of electromagnetic waves makes their interaction with conductors so huge compared with the interaction of gravity waves with matter of any kind?

16.4 ■ MOTION OF LIGHT IN MAP COORDINATES

Light reflected back and forth between mirrored test masses

LIGO is an interferometer.

The LIGO detector is an *interferometer* that employs mirrors mounted on "test masses" suspended at rest at the ends of an L-shaped vacuum cavity. The length of each leg of the L is 4 kilometers for interferometers located in the United States. Detection of the gravity wave is accomplished by measuring the difference in round-trip *time delays* between light sent down one leg of the detector and light sent down the other, perpendicular leg.

Suppose that a gravity wave of the polarization illustrated in Figure 3 moves in the z-direction as shown in Figure 5 and that one leg of the detector lies along the x-direction and the other leg along the y-direction. In order to analyze the operation of LIGO, we need to know (a) how light propagates along the x and y legs of the interferometer and (b) how the test masses at the ends of the legs move when the z-directed gravity wave passes over them. In the present section we analyze the motion of light in map coordinates; Section 16.5 begins the description of the motion of test masses in global map coordinates.

Motion of light in map coordinates.

With what map speed does light move in the x-direction in the presence of a gravity wave implied by metric (1)? To answer this question, set $dy = dz = 0$ in that equation, yielding(#SimplifiedMetric)

$$d\tau^2 = dt^2 - (1 + h)dx^2 \tag{5}$$

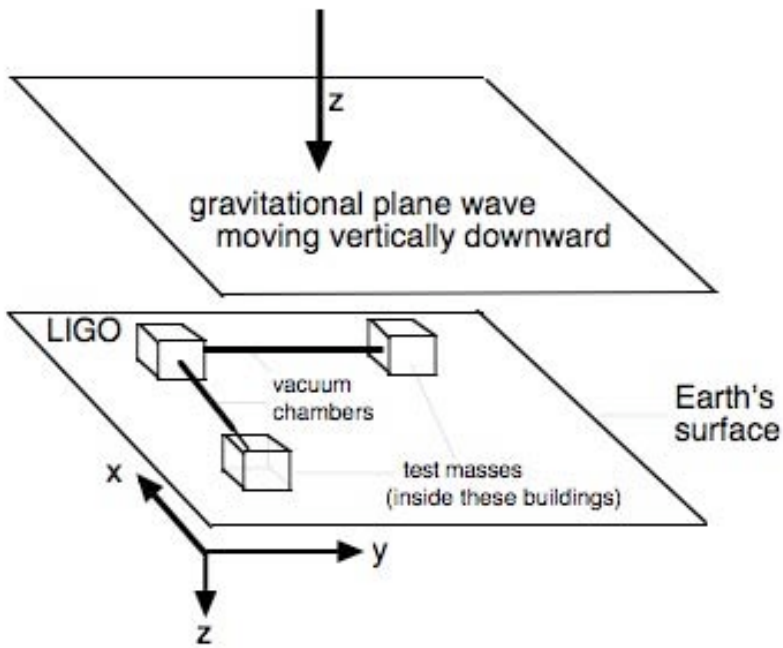

FIGURE 5 Perspective drawing of the relative orientation of legs of the LIGO interferometer lying in the x and y directions on the surface of Earth and the z-direction of the incident gravity wave descending vertically. [Illustrator: Rotate lower plate and contents CCW 90 degrees, so corner box is above the origin of the coordinate system. Same for Figure 10.]

(#LIGOSchematic)

As always, the proper time is zero between two adjacent events on the worldline of a light pulse. Set $d\tau = 0$ to find the map speed of light in the x-direction. (#LightSpeedA)

$$\frac{dx}{dt} = \pm(1+h)^{-1/2} \qquad \text{(light moving in } x \text{ direction)} \tag{6}$$

The plus and minus signs correspond to a pulse traveling in the positive or negative x-direction, respectively—that is, in the plane of LIGO in Figure 5. Remember that the magnitude of h is very much smaller than one, so we use the approximation inside the front cover. To first order: (#approx)

$$(1+\epsilon)^n \approx 1 + n\epsilon \qquad |\epsilon| \ll 1 \text{ and } |n\epsilon| \ll 1 \tag{7}$$

Apply this approximation to (6) to obtain (#LightSpeedB)

$$\frac{dx}{dt} \approx \pm(1 - \frac{h}{2}) \qquad \text{(light moving in } x \text{ direction)} \tag{8}$$

Gravity wave modifies map speed of light.

In words, the map speed of light is changed (slightly!) by the presence of our gravity wave. Since h is a function of time as well as position, the map speed of light in the x-direction is not constant, but varies as the wave passes through. (Should we worry that the speed in (8) does not have the standard value one? No! This is a *map speed*—a mythical beast—measured directly by no one.)

By similar arguments, the map speeds of light in the y and z directions for the wave described by the metric (1) are:(#LightSpeedC and LightSpeedD)

$$\frac{dy}{dt} \approx \pm(1 + \frac{h}{2}) \qquad \text{(light moving in } y \text{ direction)} \tag{9}$$

$$\frac{dz}{dt} = \pm 1 \qquad \text{(light moving in } z \text{ direction)} \tag{10}$$

16.5 ■ ZERO MOTION OF LIGO TEST MASSES IN MAP COORDINATES

"Obey the Principle of Maximal Aging!"

Consider two test masses with mirrors suspended at opposite ends of the x-leg of the detector. The signal of the interferometer due to the motion of light along this leg will be influenced only by the x-motion of the test masses due to the gravity wave. In this case the metric is the same as (5).

How does the test mass move?

How does a test mass move as the gravity wave passes over it? As always, to answer this question we use the Principle of Maximal Aging to maximize the wristwatch time of the test mass across two adjoining segments of its worldline between fixed end-events. In what follows we verify the surprising result anticipated in Section 16.2, namely that a test mass initially at rest in map coordinates rides with the expanding and contracting map coordinates drawn on the rubber sheet, so this test mass does not move with respect to map coordinates as a gravity wave passes over it. This result comes from showing that an out-and-back jog in the vertical worldline in map coordinates leads to smaller aging and therefore does not occur for a free test mass.

Idealized case: Linear jogs out and back.

Figure 6 pictures this case: an incremental linear deviation from a vertical worldline from origin 0 to the event at $t = 2t_0$. Along Segment A the displacement x increases linearly with time: $x = v_0 t$, where the speed v_0 is a constant. Along segment B the displacement returns to zero at the same constant rate. The strain h has average values $\bar{h}_{\mathrm{A}}$ and $\bar{h}_{\mathrm{B}}$ along segments A and B respectively. We use the Principle of Maximal Aging to find the value of the speed v_0 that maximizes the wristwatch time along this worldline. We will find that $v_0 = 0$. In other words, the free test mass initially at rest in map coordinates stays at rest in map coordinates; it does not deviate from the vertical worldline in Figure 6. Now for the details.

Write the metric (5) in approximate form for one of the segments: (#SimplifiedMetricC)

$$\Delta\tau^2 \approx \Delta t^2 - (1 + \bar{h})\Delta x^2 \tag{11}$$

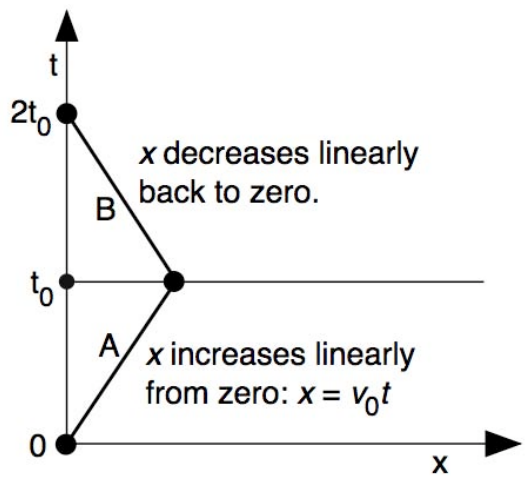

FIGURE 6 Trial worldline for a test mass; incremental departure from vertical line of a particle at rest. Segments A and B are very short.
(#GravWaveRoundTrip)

where $\bar{h}$ is an average value of the strain h across that segment. Now we apply (11) first to Segment A in Figure 6, then to Segment B. We are going to take derivatives of these expressions, which will look awkward applied to Δ symbols. Therefore we temporarily ignore the Δ symbols in (12) and let τ stand for $\Delta\tau$, t for Δt, and x for Δx, holding in mind that these symbols actually represent increments, so equations in which they appear are approximations.
With these substitutions, equation (11) becomes, for the two adjoining worldline segments:(#hoft)

$$\tau_{\mathrm{A}} \approx \left[t_0^2 - \left(1+\bar{h}_{\mathrm{A}}\right)\left(v_0 t_0\right)^2\right]^{1/2} \qquad \text{Segment A} \tag{12}$$

$$\tau_{\mathrm{B}} \approx \left[t_0^2 - \left(1+\bar{h}_{\mathrm{B}}\right)\left(v_0 t_0\right)^2\right]^{1/2} \qquad \text{Segment B}$$

so that the total wristwatch time along the bent worldline from $t = 0$ to $t = 2t_0$ is the sum of the right sides of equations (12).

We want to know what value of v_0 (the out-and-back speed of the test mass) will lead to a maximal value of the total wristwatch time. To find this, take the derivative with respect to v_0 of the sum of individual proper times and set the result equal to zero. (#intdtauD)

$$\frac{d\tau_{\mathrm{A}}}{dv_0} + \frac{d\tau_{\mathrm{B}}}{dv_0} \approx -\frac{(1+\bar{h}_{\mathrm{A}})v_0 t_0^2}{\tau_{\mathrm{A}}} - \frac{(1+\bar{h}_{\mathrm{B}})v_0 t_0^2}{\tau_{\mathrm{B}}} = 0 \tag{13}$$

so that (#intdtauE)

$$\frac{(1+\bar{h}_{\mathrm{A}})v_0 t_0^2}{\tau_{\mathrm{A}}} = -\frac{(1+\bar{h}_{\mathrm{B}})v_0 t_0^2}{\tau_{\mathrm{B}}} \tag{14}$$

Initially at rest in map coordinates? Then stays at rest in map coordinates.

Worldline segments A and B in Figure 6 are identical except in the direction of motion in x. In equation (14), v_0 is our proposed speed in global coordinates, a positive quantity. The only way that (14) can be satisfied is if $v_0 = 0$. *The test mass initially at rest does not change its map x-coordinate as the gravity wave passes over.*

Our result seems rather specialized in two senses: First, it treats only the vertical worldline in Figure 6 traced out by a test mass at rest. Second, it deals only with a very short segment of the worldline, along which $\bar{h}$ is considered to be nearly constant. Concerning the second point, you can think of (13) as a tiny out-and-back "jog" *anywhere* on a much longer vertical worldline. Then our result implies that *any* jog in the vertical worldline does not lead to an increased value of the wristwatch time, even if h varies a lot over a longer stretch of the worldline.

Not at rest in map coordinates? Maybe kink in map worldline.

The first specialization, the vertical worldline in Figure 6, is important: The gravity wave does not cause a kink in a *vertical* map worldline. The same is typically *not* true for a particle that is moving in map coordinates before the gravity wave arrives. (We say "typically" because the kink may not appear for some directions of motion of the test mass and for some polarization forms and directions of propagation of the gravity wave.) In this more general case, a kink in the worldline corresponds to a change of velocity. In other words, a passing gravity wave can change the map velocity of a moving particle just as if it were a velocity-dependent force. If the particle velocity is zero, then the force is zero: a particle at rest in map coordinates remains at rest.

QUERY 4. Disproof of relativity? (optional)

"Aha!" exclaims Kristin Burgess. "Now I can disprove relativity once and for all. If the test mass *moves*, a passing gravity wave can cause a kink in the worldline of the test mass as observed in the local inertial Earth frame. No kink appears in its worldline if the test mass is at rest. But if a worldline has a kink in it as observed in one inertial frame, it will have a kink in it as observed in all overlapping relatively-moving inertial frames. An observer in any such frame can detect this kink. So the *absence* of a kink tells me *and every other inertial observer* that the test mass is 'at rest'? We have found a way to determine absolute rest using a local experiment. Goodbye relativity!" Is Kristin right? (A detailed answer is beyond the scope of this book, but you can use some relevant generalizations drawn from what we already know to think about this paradox. As an analogy from flat-spacetime electromagnetism, think of a charged particle at rest in a purely magnetic field: The particle experiences no magnetic force. In contrast, when the same charged particle moves in the same frame, it may experience a magnetic force for some directions of motion.)

At rest in map coordinates? Still can move in Earth coordinates.

In this book we make every measurement in a local inertial frame, not using differences in global map coordinates. So of what possible use is our result that a particle at rest in global coordinates does not move in those coordinates when a gravity wave passes over it? Answer: Just because something is at rest in map coordinates does not mean that it is at rest in local inertial Earth coordinates. In the following section we find that a gravity

wave *does* move a test mass as observed in the Earth coordinates. LIGO—attached to the Earth—can detect gravity waves!

16.6 ■ DETECTION OF A GRAVITY WAVE BY LIGO

Make measurement in the local Earth frame.

Suppose that the gravity wave that satisfies metric (1) passes over the LIGO detector oriented as in Figure 5. We know how the test masses at the two ends of the legs of the detector respond to the gravity wave: they remain at rest in map coordinates (Section 16.5). We know how light propagates along both legs: as the gravity wave passes through, the map speed of light varies slightly from the value one, as given by equations (8) through (10) in Section 16.4.

Earth frame tied to LIGO slab

The trouble with map coordinates is that they are arbitrary and typically do not correspond to what an observer measures. Recall that we require all measurements to take place in a local inertial frame. So think of a local reference frame anchored to the concrete slab on which LIGO rests. (As explained in the Section 16.1, the gravity wave has essentially no effect on this slab.) Call the coordinates in the resulting local coordinate system **Earth coordinates**. Earth coordinates are analogous to shell coordinates for the Schwarzschild black hole; useful only locally but yielding the numbers that predict results of measurements. The metric for the local inertial frame then has the form: (#EarthMetric)

$$\Delta\tau^2 \approx \Delta t^2_{\text{Earth}} - \Delta x^2_{\text{Earth}} - \Delta y^2_{\text{Earth}} - \Delta z^2_{\text{Earth}} \tag{15}$$

Compare this with the approximate version of (1): (#GravWaveMetricApprox)

$$\Delta\tau^2 \approx \Delta t^2 - (1+h)\Delta x^2 - (1-h)\Delta y^2 - \Delta z^2 \qquad (h \ll 1) \tag{16}$$

Earth frame coordinate differences

Legalistically, in order to make the coefficients in (16) constants we should use the symbol $\bar{h}$, with a bar over the h, to indicate the average value of the gravity wave amplitude over the detector. However, in Query 1 you showed that for the frequencies at which LIGO is sensitive, the wavelength is very much greater than the dimensions of the detector, so the amplitude h of the gravity wave is effectively uniform across the LIGO detector. Therefore it is not necessary to take an average, and we use the symbol h without a superscript bar.

Compare (15) with (16) to yield: (#EarthCoordst,x,y,z)

$$\Delta t_{\text{Earth}} = \Delta t \tag{17}$$

$$\Delta x_{\text{Earth}} = (1+h)^{1/2}\Delta x \approx (1+\frac{h}{2})\Delta x \qquad h \ll 1 \tag{18}$$

$$\Delta y_{\text{Earth}} = (1-h)^{1/2}\Delta y \approx (1-\frac{h}{2})\Delta y \qquad h \ll 1 \tag{19}$$

$$\Delta z_{\text{Earth}} = \Delta z \tag{20}$$

where we use approximation (7). Notice, first, that Earth time lapse Δt_{Earth} between two events is identical to their map time lapse Δt and the z component of their space separation in Earth coordinates, Δz_{Earth}, is identical to the z component of their separation in map coordinates, Δz.

Now for the differences! Let Δx be the map x-coordinate separation between the pair of mirrors in the x-leg of the LIGO interferometer and Δy be the map separation between the corresponding pair of mirrors in the y-leg. As the z-directed wave passes through the LIGO detector, the test masses at rest at the ends of the legs stay at rest in map coordinates, as Section 5 showed. Therefore the value of Δx remains the same during this passage, as does the value of Δy. But the presence of the time-varying strains $h(t)$ in (18) and (19) tell us that these test masses move when observed in Earth coordinates. *More:* When the distance between test masses increases (say) along the Earth x-axis, it decreases along the perpendicular Earth y-axis; and vice versa. Perfect for detection of a gravity wave by an interferometer!

Test masses move in Earth coordinates.

Light speed $= 1$ in local Earth frame.

Earth metric (15) is that of an inertial frame in which the speed of light has the value one in whatever direction it moves. With light we have the opposite weirdness to that of the motion of test masses initially at rest: In map coordinates light moves at speeds different from unity in the presence of this gravity wave—equations (8) through (10)—but in Earth coordinates light moves with speed one. This is reminiscent of the corresponding case near a Schwarzschild black hole: In Schwarzschild map coordinates light moves at speeds different from unity, but in local inertial shell coordinates light moves at speed one.

In summary the situation is this: As the gravity wave passes over the LIGO detector, the speed of light propagating down the two legs of the detector has the usual value one as measured by the Earth observer. However, for the Earth observer the separations between the test masses along the x-leg and the y-leg change: one increases while the other decreases, as given by equations (18) and (19). The result is a difference in the round-trip times of light along the two legs. It is this difference that LIGO is designed to measure and thereby to detect the gravity wave.

Different Earth times along different legs

What will be the value of this difference in round-trip times between light propagation along the two legs? Let D be the Earth-measured length of each leg in the absence of the gravity wave. The round-trip time is twice this length divided by the speed of light, which has the value one in Earth coordinates.

Equations (18) and (19) tell us that the difference in round-trip times between light propagated along the two legs is (#OneRoundTrip)

$$\Delta t_{\text{Earth}} = 2D\left(\frac{h}{2}+\frac{h}{2}\right) = 2Dh \qquad \text{(one round trip of light)} \tag{21}$$

Time difference after N round trips.

Using the latest interferometer techniques, LIGO reflects the light back and forth down each leg approximately $N = 140$ times. That is, light executes approximately 140 round trips, which multiplies the detected time delay, increasing the sensitivity of the detector by the same factor. Equation (21) becomes (#NRoundTrips)

$$\Delta t_{\text{Earth}} = 2NDh \qquad (N \text{ round trips of light}) \tag{22}$$

Quantities N and h have no units, so the unit of time in (22) is the same as the unit of D, for example meters.

QUERY 5. LIGO fast enough?
Do the 140 round trips of light take place in a time small compared with one period of the gravity wave being detected? (If it does not, then LIGO detection is not fast enough to track the *change* in gravity strain.)

QUERY 6. Application to LIGO.
Each leg of the LIGO interferometer is of length $D = 4$ kilometers. Assume that the laser emits light of wavelength 1000 nanometer $= 10^{-6}$ meter (infrared light from a NdYAG laser). Suppose that we want LIGO to reach a sensitivity of $h = 10^{-22}$. For $N = 140$, find the corresponding value of Δt_{Earth}. Express your answer as a decimal fraction of the period T of the laser light used in the experiment.

QUERY 7. Faster derivation?
In this book we insist that global map coordinates are arbitrary human choices and do not treat map coordinate differences as measurable quantities. However, the value of h in (1) is so small that the metric differs only slightly from an inertial metric. This once, therefore, we treat map coordinates as directly measurable and ask you to redo the derivation of equations (21) and (22) using only map coordinates.

Remember that test masses initially at rest in map coordinates do not change their coordinates as the gravity wave passes over them (Section 16.4), but the gravity wave alters the map speeds of light, and differently in the x-direction, equation (8), than in the y-direction, equation (9). Assume that each leg of the interferometer has the length D_{map} in map coordinates.

A. Find an expression for the difference Δt in map time between the two legs for one round trip of the light.

B, How great do you expect the difference to be between times Δt and Δt_{Earth} and the difference between distances D (in Earth coordinates) and D_{map}? Taken together, will these differences be great enough so that the result of your prediction and that of equation (22) could be distinguished experimentally?

QUERY 8. Different directions of propagation of the gravity wave

Thus far we have assumed that the gravitational plane wave of the polarization described by equation (1) descends vertically onto the LIGO detector, as shown in Figure 5. Of course the observers cannot prearrange in what direction an incident gravity wave will move. Suppose that the wave propagates along the direction of, say, the y-leg of the interferometer, while the x-direction lies along the other leg, as before. What is the equation that replaces (22) in this case?

QUERY 9. LIGO fails to detect a gravity wave?

Think of various directions of propagation of the gravity wave pictured in Figure 3, together with different directions of x and y in equation (1) with respect to the LIGO detector. Give the name **orientation** to a given set of directions x and y—the transverse directions in (1)—plus z (the direction of propagation) in (1) relative to the LIGO detector. How many orientations are there for which LIGO will detect *no signal whatever*, even when its sensitivity is 10 times better than that needed to detect the wave arriving in the orientation shown in Figure 5? Are there zero such orientations? one? two? three? some other number less than 10? an infinite number?

16.7 ■ BINARY SYSTEM AS A SOURCE OF GRAVITY WAVES

"Newtonian" source of gravity waves

Unequal masses, each in circular orbit

Now we consider in more detail gravity waves generated by a binary system consisting of two neutron stars, each in circular orbit around their center of mass. The binary system is the only known example of a stellar system for which we can explicitly calculate the emitted gravity waves. Suppose that the stars of the binary system have masses M_1 and M_2 and are assumed to orbit at a constant distance r apart, as shown in Figure 7.

Energy of the system.

The basic parameters of the orbit are adequately computed using Newtonian mechanics, according to which the energy of the system in conventional units is given by the expression: (#Econv)

$$E_{\text{conv}} = -\frac{GM_{1,\text{kg}}M_{2,\text{kg}}}{2r} \qquad \text{(Newtonian circular orbits)} \tag{23}$$

Rate of energy loss . . .

As these neutron stars orbit, they generate gravity waves. General relativity predicts the rate at which the orbital energy is lost to this radiation. In conventional units, this rate is: (#GravRadEnergy)

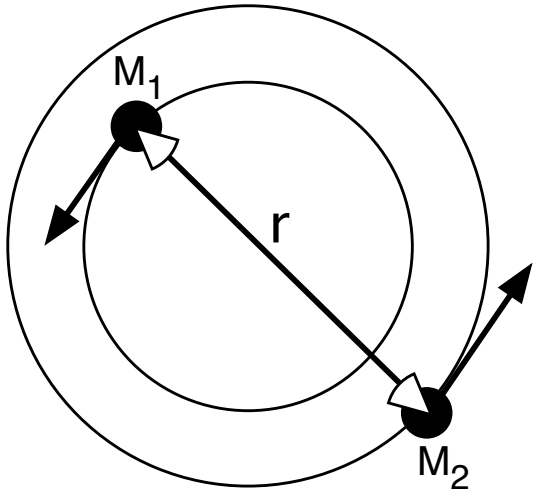

FIGURE 7 A binary system with each object in a circular path.
(#GravWaveBinary)

$$\frac{dE_{\text{conv}}}{dt_{\text{conv}}} = -\frac{32G^4}{5c^5r^5}\left(M_{1,\text{kg}}M_{2,\text{kg}}\right)^2\left(M_{1,\text{kg}}+M_{2,\text{kg}}\right) \qquad \text{(Newtonian circular orbits)} \tag{24}$$

. . . derived from Einstein's equations.

Equation (24) assumes that the two stars are separated by much more than their Schwarzschild radii and that they are moving at nonrelativistic speeds. Deriving equation (24) involves a lengthy and difficult calculation starting from Einstein's field equations. The same is true of the derivation of the metric (1) for a gravity wave. These are two of only three equations in this chapter that we simply quote from a more advanced treatment of general relativity.

QUERY 10. Energy and rate of energy loss

Convert equations (23) and (24) to units of meters to be consistent with our notation and to get rid of the constants G and c. Use the sloppy professional shortcut, "Let $G = c = 1$."

A. Show that (23) and (24) become: (#Emetric)

$$E = -\frac{M_1M_2}{2r} \qquad \text{(Newton: units of meters)} \tag{25}$$

(#GravRadEnergyB)

$$\frac{dE}{dt} = -\frac{32}{5r^5}\left(M_1M_2\right)^2\left(M_1+M_2\right) \qquad \text{(Newton: units of meters)} \tag{26}$$

B. Verify that in both of these equations E has the unit of length.

C. Suppose you are given the value of E in meters. Show how you would convert this value first to kilograms and then to joules.

QUERY 11. Rate of change of radius

Derive an expression for the rate at which the radius changes as a result of this energy loss. Show that the result is: (#RadiusChange)

$$\frac{dr}{dt} = -\frac{64}{5r^3} M_1 M_2 \left(M_1 + M_2\right) \qquad \text{(Newton: circular orbits)} \tag{27}$$

16.8 ■ BINARY PULSAR PSR1913+16

Proof that gravity waves exist?

Hulse and Taylor discover binary.

On July 2, 1974 Russell A. Hulse was carrying out observations at the world's largest radio telescope at Arecibo, Puerto Rico. Hulse—a graduate student working under the direction of Joseph H. Taylor, then at the University of Massachusetts, Amherst—detected signals from a pulsar later named PSR1913+16. (PSR stands for "pulsar" and the numbers denote its celestial coordinates.) Here is an account of the discovery, excerpted from the Nobel Foundation website (which also has wonderful illustrations) `http://www.nobel.se/physics/laureates/1993/illpres/discovery.html` (Copyright ©2001 The Nobel Foundation)

> **THE DISCOVERY OF THE BINARY PULSAR**
>
> During 1974 Joseph Taylor and Russell Hulse were searching for new pulsars with the Arecibo telescope. They discovered 40, one of which was to be very important.
>
> When Hulse was observing the new pulsar, which has been named PSR1913+16, he found that the pulses arrived sometimes more often and sometimes less. The simplest interpretation was that the pulsar was orbiting another star very closely and at high velocity: Here one "pulsar year" is only about eight hours.
>
> By observing the shift in the pulses, Hulse and Taylor found that the stars were equally heavy, each weighing about 1.4 times as much as the Sun. Since they were not visible on any photographs either, it was concluded that the other body, somewhat unexpectedly, was also a neutron star. Seen from Earth, however, it does not show up as a pulsar.
>
> . . . .
>
> **MEASURING gravity waves**
>
> Since the two neutron stars in PSR1913+16 are moving so fast and close together they should, according to General Relativity, emit large

amounts of gravity waves. This makes them lose energy: Their orbits will therefore shrink and their orbiting period will shorten.

Indirect evidence: The binary pulsar has been observed continuously since its discovery, and the orbiting period has in fact decreased. Agreement with the prediction of General Relativity is better than 0.5%. This is considered to prove that gravity waves really exist. In turn, this result is currently one of our strongest supports for the validity of the General Theory of Relativity.

Data from binary pulsar

The signal from the pulsar constituted a very stable clock, stable to 10 significant figures. As a result, Hulse and Taylor were able to use general relativity to analyze the motion of the system in detail, verifying many general relativity predictions, some of which allowed them to determine the individual orbiting masses M_1 and M_2 (given below), which Newtonian mechanics does not reveal. Their results show that the binary system PSR1913+16 has the following parameters: (#HulseParameters)

$$\begin{aligned} M_1 &= (1.442 \pm 0.003) M_{\text{Sun}} && \text{(pulsar)} \\ M_2 &= (1.386 \pm 0.003) M_{\text{Sun}} && \text{(companion)} \\ a &= 2.3418 \pm 0.0001 \quad \text{light seconds} && \text{(Semi-major axis of both)} \\ e &= 0.617127 \pm 0.000003 && \text{(Eccentricity of both)} \end{aligned} \tag{28}$$

Orbital period, ≈ 7.75 hours

Rate of advance of the periastron ≈ 4.2 degrees per Earth-year

Distance from Earth ≈ 7 kiloparsecs or about 20 000 light years. (The value of this distance has a large uncertainty.)

Meaning of *periastron*.

Each neutron star follows its own elliptical path about the center of mass. The semi-major axis of the elliptical orbit for a neutron star—label it a—is half of the major axis, the longest distance from one side of its orbit to the other. The semi-minor axis—label it b—is half of the minor axis. Then the eccentricity $e \equiv (a^2 - b^2)^{1/2}/a$. The word **periastron** refers to the point of closest approach of these "astron"omical objects (just as the word *perihelion* refers to the point of closest approach of an orbiting object to our Sun: Greek, "Helios"). Note how large the rate of this periastron advance is compared with 43 arcseconds of advance of the perihelion of the planet Mercury *per Earth-century*.

The non-zero eccentricity in equation (28) tells us that the neutron stars in PSR1913+16 are *not* in circular orbits. General relativity predicts that when a binary system has non-circular orbits it will radiate gravity waves at a greater rate than when the orbits are circular. Nevertheless, in the following Queries we assume for simplicity that the orbits are effectively circular, as in Figure 7. That is, we assume a binary system in which each companion is in a circular orbit with constant radial separation r equal to the major axis, twice

We assume circular orbits.

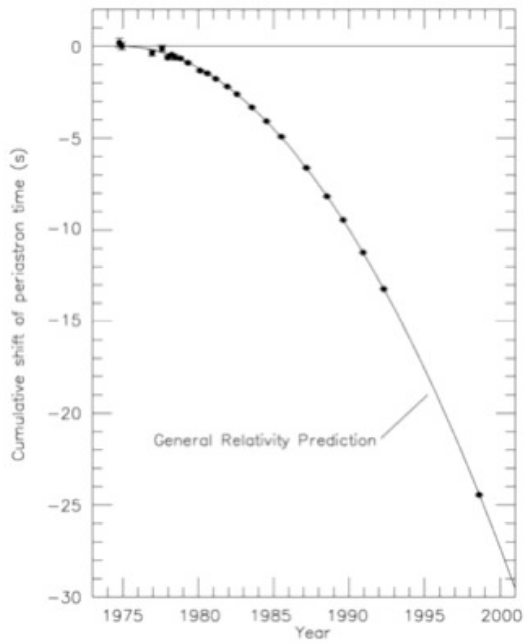

FIGURE 8 Decrease in the period in seconds (vertical axis) over the years 1975 to 1998 (horizontal axis) of binary system PSR1913+16. Agreement with the prediction of general relativity, assuming the change is due to emission of gravity waves, is now within 0.3 percent. This agreement appears to eliminate any other possible explanation for the change in orbits. From a paper (and Copyright ©2000) by J.H. Taylor and J. M. Weisberg.

(#GravWaveTiming)

the value of the semi-major axis given in (28). This is equivalent to setting to zero the eccentricity of each neutron star orbit.

QUERY 12. Shrinkage of r per orbit

For a single orbit (assumed to be circular), the separation r between the orbiting neutron stars does not change much, but it does change a little due to loss of energy to gravity waves. For one orbit, what is the approximate value of the change in this separation r? Express your answer in millimeters. (*Hint:* No integration is needed for an approximate calculation of this incremental change.)

QUERY 13. Energy radiated by idealized binary PSR1913+16

A. What is the power currently being radiated in gravity waves? Express your answer as a unitless measure (energy in meters divided by time in meters) and also in watts (joules per second).

B. Use equation (23) or (25) to calculate how much total energy in joules will be radiated in gravity waves from the present year to the future time when the two companions are separated

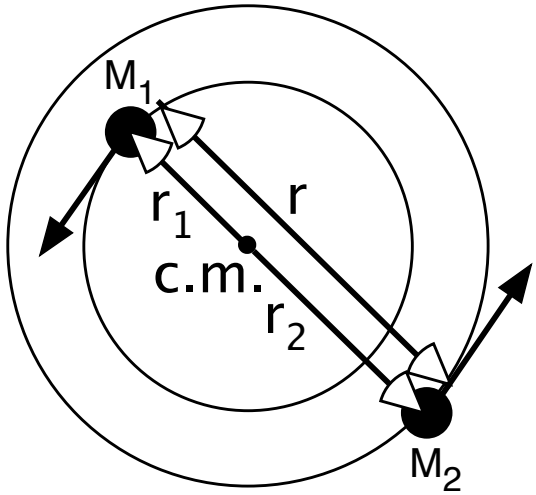

FIGURE 9 Figure 7 augmented to show the center of mass (c.m.) and orbital radii of individual masses in the binary PSR1913+16.
(#GravWaveBinaryAugmented)

by $r = 20$ kilometers (approximately the sum of their radii)? This total energy corresponds to how many kilograms of mass converted entirely to energy?

C. How long a time in years will it be before the two neutrons stars in PSR1913+16 are separated by only $r = 20$ kilometers, so that coalescence is imminent? (Only in the last millisecond or so before coalescence does the Newtonian description become completely useless.)

16.9 ■ GRAVITY WAVE AT EARTH DUE TO DISTANT BINARY SYSTEM

How far away from a binary system can we detect its emitted gravity waves?

Can LIGO on Earth's surface detect the gravity waves emitted by the distant binary system PSR1913+16 (idealized as one in which the neutron stars move in circular orbits as shown in Figure 7)? To answer this question we need to calculate the magnitude of h in the metric of equation (1).

Gravity waveform . . .

Here is the third and final result of general relativity quoted without proof in this chapter. The function $h(z,t)$ is given by the equation (in conventional units) (#hofzandt)

$$h(z,t) = -\frac{4G^2 M_1 M_2}{c^4 r z} \cos\left[\frac{2\pi f(z-ct)}{c}\right] \qquad \text{(conventional units)} \qquad (29)$$

where f is the frequency of the binary orbit, r is the (constant!) distance between orbiters in Figures 7 and 9, and z is the distance from source to detector. Convert (29) to units of meters by setting $G = c = 1$. Note that $h(z,t)$ is a function of z and t.

Figure 10 schematically displays the notation of equation (29), along with relative orientations and relative magnitudes assumed in the equation. This equation makes the Newtonian assumptions that

(a) the two stars are separated by a distance r much larger than their Schwarzschild radii, and

(b) they move at nonrelativistic speeds.

Additional assumptions are:

(c) The distance z between the binary system and Earth is very much greater than a wavelength of the gravity wave. This assumption assures that the radiation at Earth constitutes the so-called "far radiation field" where it assumes the form of a plane wave given in equation (4).

(d) The wavelength of the gravity wave is much longer than the dimensions of the LIGO detector.

(e) The binary stars are orbiting in the xy plane, so that from Earth the orbits would appear as circles if we could see them (which we cannot, because they are too far away). Unfortunately this assumption is not true of the plane of the orbit of binary PSR1913+16, as we know from Doppler shifts of signals from the orbiting pulsar.

. . . for one case

Equation (29) describes only one linear polarization at Earth, the one generated by metric (1) and shown in Figure 3. The orthogonal polarization shown in Figure 4 is also transverse and equally strong, with components proportional to $(1 \pm h)$. The formula for the magnitude of h in that orthogonally polarized wave is identical to (29) with a sine function replacing the cosine function. We have not displayed the metric for that orthogonal polarization.

Detection requirements

In order for LIGO to detect a gravity wave, two conditions must be met: (a) the amplitude h of the gravity wave must be sufficiently large, and (b) the frequency of the wave must be in the range in which LIGO is most sensitive (100 to 400 hertz). Query 14 deals with the amplitude of the wave. The frequency of gravity waves, discussed in Query 15, contains a surprise.

QUERY 14. Amplitude of gravity wave from PSR1913+16 at Earth

A. Use (29) to calculate the maximum amplitude of h at Earth due to the radiation from the "idealized circular-orbit" binary system PSR1913+16. Consider this amplitude to be positive.

B. Can either Initial LIGO or Advanced LIGO detect the gravity waves whose amplitude is given in part A?

C. What is the maximum amplitude of h at Earth just before coalescence of PSR1913+16, when the neutron stars are separated by a distance $r = 20$ kilometers (but with orbits still described approximately by Newtonian mechanics)?

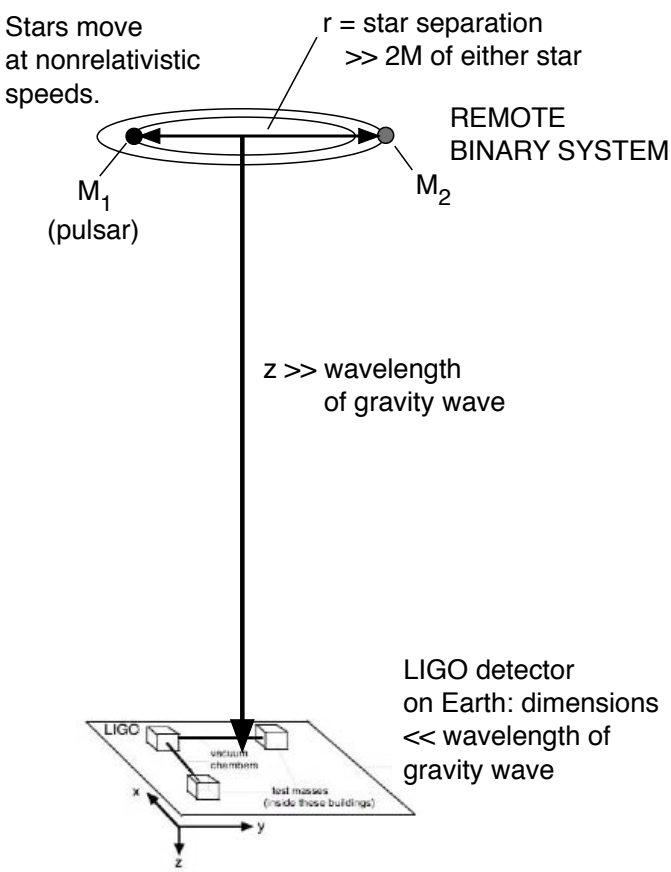

FIGURE 10 Schematic diagram, *not to scale*, showing notation and relative magnitudes for equation (29). The binary system and the LIGO detector lie in parallel planes.[Illustrator: See note in caption to Figure 5.]

(#GravWaveCylinder)

QUERY 15. Frequency of gravity waves emitted from PSR1913+16

A. In order for either Initial LIGO or Advanced LIGO to detect the gravity waves whose amplitude is given in Query 14, the frequency of the gravity wave must be in the range 100 to 400 hertz. In Figure 9 the point C. M. is the stationary center of mass of the pulsar system. Using the symbols in Figure 9, fill in the steps to complete the following derivation. (#freqA) (#freqB) (#freqC) (#freqD)

$$\frac{v_1^2}{r_1} = \frac{GM_1}{r_1^2} \qquad \text{(for } M_1\text{, Newton, conventional units)} \tag{30}$$

$$\frac{v_2^2}{r_1} = \frac{GM_2}{r_2^2} \qquad \text{(for } M_2\text{, Newton, conventional units)} \tag{31}$$

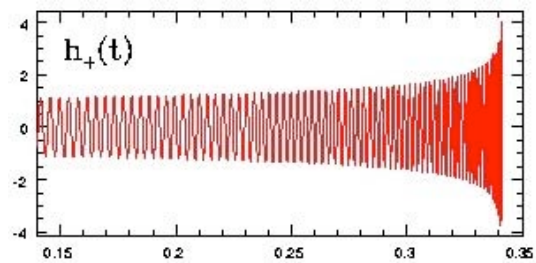

FIGURE 11

Predicted "chirp" of one polarization of gravity wave as the two components of a binary system coalesce. (The plot for the other polarization is indistinguishable from this one.) (#GravWaveChirp) (#HORIZONTAL AXIS IN SECONDS?)

$$M_1 r_1 = M_2 r_2 \qquad \text{(center-of-mass condition)} \tag{32}$$

$$f_{\text{orbit}} \equiv \frac{1}{T_{\text{orbit}}} = \frac{v_1}{2\pi r_1} = \frac{v_2}{2\pi r_2} \qquad \text{(common orbital frequency)} \tag{33}$$

where f_{orbit} and T_{orbit} are the frequency and period of the orbit, respectively. From these equations, show that for $r \equiv r_1 + r_2$ the frequency of the orbit is (#freqE)

$$f_{\text{orbit}} = \frac{1}{2\pi}\left[\frac{G\,(M_1 + M_2)}{r^3}\right]^{1/2} \tag{34}$$

B. Here is a surprise: The frequency f of the gravity wave generated by this binary pair and appearing in (29) is twice the orbital frequency. (#freqF)

$$f_{\text{gravity wave}} = 2 f_{\text{orbit}} \tag{35}$$

Why this doubling? Essentially it is because gravity waves are waves of tides. Just as there are two high tides and two low tides per day caused by the moon's gravity acting on the Earth, there are two peaks and two troughs of gravity waves generated per binary orbit.

C. Approximate the average of the component masses in (28) by the value $M = 1.4 M_{\text{Sun}}$. Find the distance r between the binary stars when the orbital frequency is 75 hertz, so that the frequency of the gravity wave is 150 hertz. [ANS: Approximately 100 km.]

D. Using results quoted earlier in this chapter, estimate the time for the binary system to decay from the current radial separation to the radial separation calculated in part C.
ANS: $t_2 - t_1 = 5(r_2^4 - r_1^4)/(256 M^3)$, everything in unit meter.

"Chirp" at coalescence

Newtonian mechanics predicts the motion of the binary system surprisingly accurately until the two components touch, a few milliseconds before they coalescence. Newton tells us that as the separation r between the

orbiting masses decreases, their orbiting frequency increases. As a result the gravity wave sweeps upward in both frequency and amplitude in what is called a **chirp**. Figure 11 is a predicted wave form for such a chirp. (To hear an audio simulation of the chirp, search for "gravity wave chirp of coalescing neutron stars" in your browser.)

Detection of such a waveform sweeping through the frequencies for which LIGO is sensitive would be a "smoking gun" for the coalescence of a binary source. Although LIGO cannot detect emission from PSR1913+16, we expect that many other binary systems are close enough to provide a detectable signal for Advanced LIGO.

16.10 ■ REFERENCES

Initial quote: Marcia Bertusiak, *Einstein's Unfinished Symphony: Listening to the Sounds of Space-Time*, 2000, Washington D.C., Joseph Henry Press.

LIGO sensitivity, Figure 2, at
`http://www.ligo.caltech.edu/advLIGO/scripts/summary.shtml`

Chirp wave shape, Figure 11, at
`http://www.lsc-group.phys.uwm.edu/~patrick/work/talks/itp/itp0008.gif`

Websites for updates:
`www.ligo.org`
`www.ligo.caltech.edu`