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Gravity Research Foundation
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Ladies and Gentlemen:

Enclosed find three copies of the Essay entitled "Closed Universes: Their Future Evolution and Final State", by John D. Barrow and myself, which we are submitting for the 1985 Essay Contest. Also included are three copies of a short summary, and biographical information on Dr. Barrow and myself.

Sincerely yours,

Frank J. Tipler

Frank J. Tipler

CLOSED UNIVERSES: THEIR FUTURE EVOLUTION AND FINAL STATE

Summary

We summarize what is currently known about the future evolution and final state of closed universes: in mathematical language, those which have a compact Cauchy surface. We show that the existence of a maximal hypersurface (a time of maximum expansion) is a necessary and sufficient condition for the existence of an all-encompassing final singularity in a universe with a compact Cauchy surface. Not all closed universes can admit a maximal hypersurface, but we state a theorem giving a complete classification of those closed universes which do. The relevance of these results to inflation is also discussed.

Frank J. Tipler: Biographical Information

F.J. Tipler is an associate professor of mathematical physics at Tulane University. Born in 1947, he graduated with a B.S. in physics from the Massachusetts Institute of Technology in 1969. In 1976 he received a Ph.D. in physics from the University of Maryland under the supervision of the well-known relativist, Dieter R. Brill. After receiving his doctorate, he spent three years as a post-doctoral fellow in the mathematics department at the University of California at Berkeley. At the University of California he was associated with A.H. Tuab's research group in relativity. He held an S.R.C fellowship at Oxford University in D.W. Sciama's group during the summer of 1979. From 1979 to 1981 he was a postdoctoral fellow in John A. Wheeler's group at the University of Texas at Austin. In the Fall of 1981 he moved to Tulane University as an Associate Professor.

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J.D. Barrow is a university lecturer in Astronomy at the University of Sussex, U.K. Born in 1952, he graduated with 1st Class Honours in Mathematics from the University of Durham in 1974 receiving the University Prize and the Collingwood Prize of the London Mathematical Society. In 1977 he received a doctorate in Astrophysics from Oxford University under the supervision of the well-known cosmologist D.W. Sciama. For this research work he was awarded the Johnson Memorial Prize and the Wallace Research Prize by Oxford University and Magdalen College, Oxford. Prior to completing his doctoral studies he was elected a Junior Research Lecturer at Christ Church, Oxford to become the youngest senior member of the University and was awarded the Lindemann Travelling Fellowship by the English Speaking Union of the Commonwealth. He held the Lindemann Fellowship in the Astronomy Department of the University of California at Berkeley during 1977/8 before returning to Christ Church and the Department of Astrophysics, Oxford until 1980 when he was awarded a Miller Fellowship by the University of California at Berkeley. This was held in the Department of Physics until he took up a permanent lectureship at the Astronomy Centre, Sussex University in 1981.

CLOSED UNIVERSES: THEIR FUTURE EVOLUTION AND FINAL STATE

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1 Introduction

Zel'dovich and Grishchuk (1984) have recently argued that the spontaneous formation of the universe from "nothing" requires the universe to have compact spacelike sections, i.e., to be closed. Moreover, Hawking (1984a,b) and Tipler (1984) have argued that a consistent quantum gravity theory can be formulated only for closed universes. However, current observations seem to show that the density parameter Ω_0 is less than 0.3 in that portion of the Universe visible to us. These observations could nevertheless be consistent with compactness of the Universes's spatial sections if the sections had a non-spherical topology (for instance a three-torus T^3), or a spherical (S^3) topology in which the under-density in our region was counterbalanced by an over-density elsewhere. Zel'dovich and Grishchuk (1984) investigate the future evolution of a spherically symmetric but inhomogeneous cosmology with S^3 spatial topology, and they show that this model recollapses to an all-encompassing final singularity as does the more familiar Friedman universe with three-sphere spatial topology. They conjecture that this recollapse to a final singularity is a generic property of closed universes, and that a proof of this conjecture along the lines of the Hawking-Penrose singularity theorems can probably be found.

There have been a number of theorems proven in global general relativity in recent years which have a bearing on this conjecture. In this 1985 gravity award essay we summarize what is known. In brief, the theorems indicate the following. First, a necessary and sufficient condition for both a final and initial all-encompassing singularity in a globally hyperbolic closed universe satisfying the strong energy condition and a generic condition is the existence of a maximal hypersurface, or a time of maximum expansion. Second, globally hyperbolic closed universes with a complicated spatial topology (for instance T^3) cannot have a maximal hypersurface and hence cannot recollapse. On the other hand, all known globally hyperbolic closed universes with S^3 spatial topology do in fact have maximal hypersurfaces and hence recollapse to final all-encompassing singularity. The Zel'dovich-Grishchuk model is a very interesting additional example of the recollapse of three-sphere closed universes. On the basis of the known examples, Marsden & Tipler (1980) conjectured that all globally hyperbolic three-sphere universes satisfying the strong energy condition and a generic condition have a maximal hypersurface and hence recollapse to an all-encompassing final singularity.

Surprisingly, not much progress has been made in proving a general theorem to this effect. The Zel'dovich-Grishchuk result, that all spherically symmetric three-sphere universes with positive matter density and zero cosmological constant recollapse, is actually the most general result known. We shall discuss the efforts to prove a more general theorem in detail in section 2.

Zel'dovich and Grishchuk (1984) also suggest that the final singularity, although all-encompassing, may not be simultaneous.

Global simultaneity, of course, is a concept which depends on a choice of a global coordinate system. There is a growing consensus among relativists that the most natural global coordinate system is the one defined by the foliation (slicing of the space-time by hypersurfaces[1]) of constant mean curvature hypersurfaces. We shall show that under fairly general conditions such a foliation will exist, and if the curvature grows sufficiently fast near the final singularity, the final singularity will be simultaneous in the time coordinate defined by this foliation.

We shall state the theorems in the precise language of global general relativity, but we shall also try to explain the physical meaning of the terms employed so as to make these results understandable to those not conversant with this rather esoteric language; for further discussion see Barrow and Tipler (1985). Our notation and conventions will be those of Hawking & Ellis (1973) except that we do not set $G \equiv 1$. In particular, R_{abcd} is the Riemann curvature tensor, R_{ab} is the Ricci tensor, and the metric signature is (- + + +).

2 The Global Theorems

Our first theorem establishes the necessity of all-encompassing initial and final singularities if a closed universe is globally hyperbolic and has a maximal hypersurface. A space-like hypersurface S is said to be a **maximal hypersurface** if $z^a{}_{;a} = 0$ everywhere

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[1]As an example, the surface of an ordinary cylinder is a two-dimensional surface that can be foliated by a sequence of circles perpendicular to its axis. The cylinder is generated by all of these circles stacked on top of each other. The stack is the 'foliation' of the cylinder by circles.

on S, where z^a is the unit normal vector to S.

THEOREM 1: The length of <u>every</u> timelike curve in a space-time (M,g) is less than a universal constant L if the following conditions hold:

(1) $R_{ab}k^a k^b \geq 0$, for all timelike vectors k^a;

(2) there exists a compact maximal hypersurface S;

(3) the space-time is globally hyperbolic;

(4) at least one of the tensors $z^c z^d z_{[a}R_{b]cd[e}z_{f]}$, $z_{a;b}$, or $R_{ab}z^a z^b$, is non-zero somewhere on the maximal hypersurface S, where z^a is the normal vector to S.

This theorem was first explicitly stated and proved by Marsden and Tipler (1980). Since the length of every timelike curve in the space-time is less than L, this means that all timelike curves must begin at an initial singularity, and terminate at a final singularity. The entire space-time can be shown to have a finite four-volume, so a cosmology satisfying the conditions of Theorem 1 is closed in both space and time. The length function d(p,q) between two events p and q is defined by Hawking & Ellis (1973) to be the length of the longest timelike curve connecting them. Theorem 1 shows that $d(M,M) < L$, where M is the space-time manifold. Tipler (1977), and Marsden and Tipler (1980) termed a space-time with a compact Cauchy surface and $d(M,M) < L$ a **Wheeler Universe**.

Hawking & Ellis (1973) call condition (1) of Theorem 1 the **timelike convergence condition**. If the Einstein equations hold, it is implied by the **strong energy condition**, which says that the stress-energy tensor T_{ab} satisfies $T_{ab}z^a z^b - (1/2)T^a{}_a z^b z_b \geq (1/8\pi G)\Lambda$, where Λ is the cosmological constant.

Condition (3) says that if initial data is given on a compact spacelike hypersurface, then the entire future and past of this hypersurface is determined by this initial data and the evolution equations. Such a hypersurface is called a **Cauchy Surface**[2]. Budic <u>et al</u> (1978) have shown that in a space-time with a compact Cauchy surface, all compact spacelike hypersurfaces are in fact Cauchy surfaces.

Condition (4) says that somewhere on the maximal hypersurface, the gravitational forces are non-zero, or at least the maximal hypersurface is not a surface of time symmetry. If the gravitational forces due to matter are non-zero, the scalar $R_{ab}z^a z^b$ will be non-zero, and if the gravitational tidal forces are non-zero along timelike curves normal to the hypersurface, the tensor $z^c z^d z_{[a}R_{b]cd[e}z_{f]}$ will be non-zero. For vacuum space-times, the condition $z_{a;b} = 0$ with z^a the unit normal to a spacelike Cauchy surface S will imply that the future and past of S are identical. Thus, such a hypersurface is called a **surface of time symmetry**. In any physically realistic space-time, one which has some irregularity, we would expect condition (4) to

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[2]The precise definition of a Cauchy surface is that it is a hypersurface which every timelike curve intersects exactly once, but if the initial value problem is well-posed, this definition is equivalent to the one above. Global hyperbolicity precisely stated says that all sets of the form $J^+(p) \cap J^-(q)$ are compact, but this definition is actually equivalent to the existence of a Cauchy surface; see Hawking & Ellis (1973) for a discussion.

hold; we can think of condition (4) as a 'generic condition' to eliminate pathological examples which are unphysical.

Condition (4) is necessary condition for Theorem 1 to hold. For example, the Einstein static universe satisfies all the conditions in the theorem except (4), and of course the Einstein static universe is singularity free. But the Einstein universe is unstable.

Theorem 1 shows that the existence of a compact maximal hypersurface is a sufficient condition for a space-time to have an initial and a final all-encompassing singularity. The next theorem shows that it is a necessary condition.

THEOREM 2: If (M,g) is a Wheeler universe, then it has a unique maximal spacelike C^2 hypersurface, and it can be foliated uniquely by constant mean curvature hypersurfaces, with the mean curvature varying from $+\infty$ at the initial singularity to $-\infty$ at the final singularity if the following conditions hold:

(1) $R_{ab}k^a k^b \geq 0$ for all timelike vectors k^a;

(2) the region of (M,g) near the initial and final singularities can be foliated by Cauchy surfaces S(t), S'(t) respectively which satisfy

$$\max_{p \in S(t)} z^a{}_{;a}(p) \to +\infty$$

as S(t) approaches the initial singularity, and

$$\max_{p \in S'(t)} z^a{}_{;a}(p) \to -\infty$$

as S'(t) approaches the final singularity.

Condition (2) of Theorem 2 is a restriction on the nature of the initial and final singularities, and such a restriction is essential if one is to obtain the existence of a maximal hypersurface. The reason is that in global general relativity, a singularity is indicated by the presence of an incomplete causal curve, and incompleteness of causal curves can result from both true infinite curvature singularities, and also "singularities" which arise from not extending the space-time as far as one could. For example, we can construct a Wheeler universe from the flat Friedman universe by identifying the spatial points $x = 0$, $x = 1$; the points $y = 0$, $y = 1$; and the points $z = 0$, $z = 1$ to transform the spacelike Cauchy surfaces with topology R^3 into Cauchy surfaces with topology T^3, and then cutting away the region to the future of the hypersurface $t = 1$ inclusive, and to the past of the hypersurface $t = 0$ inclusive. The resulting space-time will exist for only one time unit, beginning at the "singularity" $t = 0$, and ending at the "singularity" $t = 1$. Such a Wheeler universe has no maximal spacelike hypersurface.

However, we would expect that a real singularity would act to decrease the size of the universe near the singularity, and this intuitive idea of universal contraction is expressed by condition (2). Locally, the trace of the tensor $z_{a;b}$, which is the second fundamental form, or extrinsic curvature, of the hypersurface S, can be written as

$$z^a{}_{;a} = (1/dV)d(dV)/dt \qquad (1)$$

where dV is the local infinitesimal volume element (Marsden

& Tipler (1980)). Thus if the universe is contracting everywhere to zero volume, the right-hand-side of (1) will go to plus infinity at the initial singularity, and minus infinity at the final singularity. Eardley and Smarr (1979) introduced condition (2); they termed a singularity which satisfies condition (2) a **crushing singularity**. The existence of a foliation of the space-time by Cauchy surfaces near the singularities, which is required in the definition of a crushing singularity, actually follows from global hyperbolicity, and is not an additional assumption.

There are other ways of expressing the idea that the initial and final singularities crush objects out of existence and are not just artificial singularities. A discussion of these alternatives, which also yield an existence theorem for maximal hypersurfaces in Wheeler universes, can be found in Marsden & Tipler (1980), and in Tipler, Clarke & Ellis (1980).

The proof of Theorem 2 is due to a number of people. Marsden & Tipler (1980) obtained the existence of a Lipshitz compact Cauchy surface of maximal volume under the hypotheses of Theorem 2, but they were unable to show that this hypersurface was everywhere spacelike and C^2; the possibility existed that the maximal volume hypersurface possessed kinks and null portions. Marsden and Tipler also showed that if the possibility of the leaves of a constant mean curvature foliation turning null were eliminated, then such a foliation existed. These possibilities were eliminated by Gehardt (1983), who was able to prove the existence of a

C^2 maximal spacelike hypersurface and a foliation of constant mean curvature hypersurfaces under the conditions of Theorem 2. Bartnik (1984) has recently obtained a simpler proof. Brill and Flaherty (1976) had earlier proved that a constant mean curvature foliation, if it existed, was unique.

Thus the existence of a maximal spacelike hypersurface is a necessary and sufficient condition for the existence of all-encompassing initial and final singularities. The next theorem shows that a maximal hypersurface will never evolve in some universes with compact Cauchy surfaces; the existence of a maximal hypersurface is permitted only in closed universes with rather special spatial topologies.

THEOREM 3: If S is a spacelike compact orientable maximal hypersurface, then it must have topology

$$(S^3/P_1) \# (S^3/P_2) \# \ldots \# (S^3/P_n) \# k(S^2 \times S^1) \qquad (2)$$

where P_i is a finite subgroup of SO(3), "#" denotes the connected sum, and $k(S^2 \times S^1)$ means the connected sum of k copies of $S^2 \times S^1$, provided the following conditions hold:

(1) The Einstein equations without cosmological constant hold on the spacetime (M,g);

(2) The weak energy condition holds;

(3) the induced metric on S is not flat;

(4) The differentiable structure on $M = S \times R^1$ is not exotic.

We remind the reader that, roughly speaking, a **connected sum** of two three-dimensional manifolds is the manifold formed by cutting a small spherical volume out of each, and then gluing the two manifolds together along the boundaries of the remaining manifolds. A detailed discussion of the connected sum can be found in any book on advanced topology; e.g., Hempel (1976). The **quotient** S^3/P_i of a three-sphere S^3 with a subgroup P_i means identifying points of the three-sphere which are carried into one another under the action of the subgroup. The **weak energy condition** means that the stress-energy tensor satisfies $T_{ab}k^ak^b \geq 0$. A **non-exotic differentiable structure** on $M = S \times R^1$ means that the coordinate systems covering M are generated by pulling up the coordinate systems which cover S. See Freedman (1982), Donaldson (1983), and especially Freed & Uhlenbeck (1984) for a discussion of exotic differentiable structures on four-manifolds.

Theorem 3 is a re-expression of a theorem of Schoen & Yau (1978, 1979a&b). They showed that the only non-flat 3-manifolds to admit a metric with scalar curvature ${}^{(3)}R$ non-negative have topology (2). (More precisely, they showed that the 3-manifolds had to have the form (2) but with each S^3/P_i replaced with $\underline{S}^3/P_i$, where $\underline{S}^3$ is a homotopy sphere. S.-T. Yau has pointed out to us, however, that if we assume the differentiable structure of $M = S \times R^1$ to be non-exotic, then homotopy spheres must be spheres. It is possible that condition (4) is not necessary.) The initial value Einstein equations without cosmological constant constrain ${}^{(3)}R$ in S (Misner, Thorne & Wheeler (1973)) as follows:

$$ {}^{(3)}R + (z^a{}_{;a})^2 = 16\pi G T_{ab}z^a z^b + z_{a;b}z^{a;b} \qquad (3) $$

where z^a is the unit normal to the spacelike hypersurface S.

On a maximal hypersurface, $z^a{}_{;a} = 0$, and on any hypersurface $z_{a;b}z^{a;b} \geq 0$, since the second fundamental form has only spacelike components in Gaussian normal coordinates. Thus if conditions (1) and (2) of Theorem 3 hold, $^{(3)}R \geq 0$ on a maximal hypersurface, so the only possible topologies for maximal hypersurfaces are given by (2).

In particular, closed universes with spatial topology T^3 cannot have maximal hypersurfaces and thus cannot re-collapse to a final singularity. The three-torus universe we constructed above by identifying the flat Friedmann universe is an example: its maximal extension in time will begin at an initial curvature singularity, but it will expand forever. Brill (1977) was the first to conjecture that T^3 universes must expand forever.

On the other hand, universes with S^3 Cauchy surfaces can in principle admit maximal hypersurfaces and may re-collapse. In fact, as was discussed by Marsden & Tipler (1980), all known examples of cosmologies with three-sphere Cauchy surfaces, which satisfy a generic condition and the timelike convergence condition, begin and end in all-encompassing singularities. This led Marsden & Tipler (1980) to conjecture that all such cosmologies begin and end in singularities. The Zel'dovich & Grishchuk spherically symmetric but inhomogeneous model provides more evidence for this conjecture.

It is not known if the general homogeneous three-sphere cosmologies, the Bianchi type IX universes, all recollapse. Both Matzner, Shepley & Warren (1970) and S.P. Novikov (1972) have published proofs which purport to show that all Bianchi type IX universes recollapse, but unfortunately their proofs are incomplete[3].

Schoen & Yau have shown (1979b) that all oriented 3-manifolds with topology (2), but where the finite subgroup P_i acts linearly, actually admit spacelike metrics with $^{(3)}R > 0$. Thus except for the restriction that P_i act linearly, they have shown in effect that for every topology of the form (2), there exist globally hyperbolic solutions to the Einstein equations in which the Cauchy surfaces have the given topology, and the maximal developments begin and end in singularities. Yau conjectures that the hypotheses of Theorem 3 are sufficient to insure that P_i must act linearly, but he has no proof. As with the S^3 case, it is not known if all solutions with the given Cauchy surface topology must begin and end in singularities.

3 Astrophysical Applications

The motivation for Zel'dovich and Grishchuk's original discussion of closed universes was a prediction of inflationary universe models: the Cosmological Principle should hold locally yet globally there should exist considerable inhomogeneity, perhaps

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[3]The Matzner et al proof introduces a $\delta > 0$, and δ must be bounded away from zero in order for the proof to go through, but the possibility that δ could approach zero is not eliminated. The Novikov proof identifies the reaching of the maximal hypersurface with the vanishing of an "energy" U. Novikov shows that dU/dt ≤ 0, but does not eliminate the possibility that the dU/dt could vanish so rapidly that U = 0 is never reached. The proof is non-trivial because the three-curvature of the Bianchi type IX universe can be either positive or negative at different stages of its evolution; in fact it is only positive when the model is close to isotropy.

on a scale exceeding our present particle horizon. If the Cosmological Principle is guarenteed out to a scale $\sim H_0^{-1}$ (where H_0 is the present Hubble parameter) then the present density will be within one part in 10^6 of the critical density required for closure (Guth (1981)). However, inflationary universes may give rise to a minimum amplitude of density inhomogeneity that exceeds 10^{-6} (Gibbons et al, 1983). Thus, small density inhomogeneities over large scales may have less than the critical density even though the global mean density exceeds the critical value. Can these low density regions escape the future singularity? This is the question answered by Zel'dovich & Grishchuk for a particular model and analyzed by ourselves above. We note, in passing, that there exists tentative observational evidence for large regions of under-density in the Universe (Kirshner et al (1981); Gregory & Thompson (1978); see however Geller, (1985)). Such regions should also leave fluctuations in the microwave background temperature profile over small angular scales.

The problem of interpreting the evidence for and against the existence of extensive void regions is compounded by the fact that the bulk of gravitating matter in the Universe is in non-luminous form. The universal mass distribution could, in principle, be dominated by objects varying in mass from 10^{-38} gm (axions) to 10^{39} gm (supermassive black holes). Nevertheless, it is possible to resolve virial mass discrepancies in clusters and explain the form of rotation curves in spirals with a density parameter $\Omega_0 \leq 0.2$. Furthermore, the random motion of galaxies place a limit of $\Omega_0 \leq 0.24$ on all non-uniformly clustered material (Bean et al, 1983). Thus taken at face value the evidence provides support neither for the inflationary prediction of $|\Omega_0 - 1| < 10^{-6}$, nor for a closed universe with $\Omega_0 > 1$. In order to reconcile observation with either of these possibilities there must be either a non-zero cosmological constant (Peebles 1984), or a smooth distribution of unfound cosmic material not clustered with the luminous matter (for example, new weakly interacting elementary particles or photons arising from elementary particle decays at a recent redshift). If inflation occurs and drives the geometry of space very close to Euclidean and the effective cosmological constant dominating the expansion during the inflationary phase is not completely cancelled then in a Friedman universe we would now observe

$$\Omega_0 = 1 - \frac{\Lambda}{3H_0^2} \qquad (3)$$

Thus, if $\Lambda > 0$, we can reconcile inflation with observations of $\Omega_0 < 1$. However, this is an extremely unattractive remedy since the motivation for inflation was to remove assumptions about extraordinary fine-tuning of initial conditions, whereas (3) requires the residual Λ term to be very specially chosen, i.e., $\Lambda G \leq 10^{-120}$ in dimensionless units. If a residual positive cosmological constant exists, then the Universe will expand forever, whatever its spatial topology. If a residual negative cosmological constant exists, then all globally hyperbolic universes recollapse if the stress-energy tensor obeys the strong energy condition (Tipler, 1976).

The first two theorems given in section 2 rely on the strong energy condition, which can be violated even with normal matter if the cosmological constant is positive. If $\Lambda = 0$, then it is still possible that this condition will be violated near the final singularity in a closed universe when the temperature and hence the particle energy becomes very high. The complicated self-interaction potentials expected in the Higgs sector of grand unified gauge theories typically lead to stress tensors that violate the strong energy condition (although not the weak energy condition) at high energy (Tipler, 1978). For example, a simple scalar field with self-interaction potential $V(\phi) = \lambda\phi^4$, $0 < \lambda << 1$, employed in the chaotic inflation models of Linde (1984) leads to a breakdown of the strong energy condition. If such a breakdown occurs, an S^3 closed universe would recollapse, but it would not recollapse to a final singularity.

In conclusion, we see that the question of the global structure of the Universe rests upon two aspects which are not predicted by inflationary theories: the spatial topology and the sign of the cosmological constant.

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