Qualitative Aspects Of Non-Equilibrium Statistical Distributions For Turbulent Flows

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Abstract: This paper puts forward the key features of plausible time-evolving statistical distributions of the Scale and Spin of 'eddies', describing the phenomenon of turbulence. Statistical distributions generally describe the state of the system in equilibrium states, but describing turbulent flows demands non-equilibrium statistics, as there is energy dissipation due to viscosity of the fluid and variations in macroscopic properties of the system with time. The approach put forward proves to be highly advantageous to explain the variation of eddy scale, using the 'Scale Displacement Equation'. It ascertains a huge gain in understanding characteristics of the fluids subject to high Reynolds numbers. The phenomenon of 'Spin Interference', which is responsible for the variation of eddy spin in the system, is also described intensively. The put forth Scale and Spin statistics, parametrized by time, demonstrate some interesting features that are fundamentally linked to the behaviour of fluids.

Index Terms: Turbulence; Kolmogorov Microscales; Poisson Distribution; Non-Equilibrium Statistics; Scale Displacement Equation; Construction and Destruction Coefficients; Spin Interference Equation.

1 INTRODUCTUCTION

Turbulence is a complex phenomenon, complex at every scale. Just like velocities of individual gas molecules are understood on a statistical basis, turbulence is also described through statistical physics. But, such a description for turbulence leads to inconsistencies. One of them is self-dissimilarity, which signifies that the distributions of macroscopic properties of fluid flow are not scale invariant. Due to the scale dependence, statistics of both scale and spin need to be coupled together in order to bring out an accurate picture of the phenomenon. Generally, the main structure of turbulent fluids is an eddy. As, the Reynolds Number crosses four thousand, eddies begin to form. These eddies propagate through the fluid, divide into smaller eddies and dissipate energy when their size approaches the Kolmogorov Length Scale. This scale is defined as,

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$$

Here, ν is the kinematic viscosity of the fluid and ϵ is the rate of energy dissipation. At this scale, the viscosity of the fluid overcomes the rotational energy of the eddies. This energy is converted to internal energy of the fluid. In isotropic turbulence, the larger eddies transfer energy to lower scaled eddies, and this process continues till the attainment of equilibrium, when there is no extra source of energy in the system. Due to the non-equilibrium states of the time evolving turbulent system, the average parameters of the fluid change. The most probable eddy scale decreases with time. Yet, spin direction is conserved for an isolated eddy but decreases in magnitude over time. Let us discuss the aspects of the Scale Distribution.

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2 THE SCALE DISTRIBUTION

Firstly, we assume that, the medium is isotropic, and the external source of energy (if any) is also isotropic. By asymptotic analysis and observation of the eddies, It is known that, eddies larger than the most probable scale, are rarer to be found in the system. Thus the fraction of eddies having a higher scale approaches zero. Now, by Kolmogorov Theory, the rotational energy of an eddy is converted to thermal energy only below the Kolmogorov scale. So the fraction of total eddies, having scale smaller than the Kolmogorov microscale is zero. Thus, the probability density function approaches zero at two points, namely, the Kolmogorov length scale η and infinity. Due to division of eddies under cascading process, the most probable eddy scale of the system decays exponentially. The decay rate depends on the instantaneous size of the eddy. The eddies being considered here are rotationally symmetric. This implies these are not subject to skewing action caused due to macroscopic flow. Thus, scale of an eddy is essentially its radius. Eddy division is a pseudo-random process. It is governed by simple laws, still it shows chaotic behaviour. Also, the distribution might have a positive or a negative skewness coefficient, depending upon the system.

3 THE SCALE DISPLACEMENT EQUATION

It is known that the eddies form a hierarchy in the system. The larger eddies decompose into smaller ones. Over time the most probable length scale to be found in the system decreases in absence of an external energy source. It happens when a turbulence inducing impulse is transmitted into the fluid. The time between two decays of an eddy is controlled by its viscosity. When viscosity is large, the viscous forces dominate, and the eddy is held more effectively. Thus to model this displacement of the most plausible length scale, we need to inculcate the exponential function. Let, the most probable scale be (α+1) times the Kolmogorov length scale. Thus,

$$\eta_{mp} = (\alpha + 1)\eta = (\alpha + 1)\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$$

Now, let τ be the time constant for eddy division. This is parametrized by the viscosity of the fluid and controls the time interval between two successive eddy divisions. Thus,

$$\tau = f(\nu)$$

So, the peak or the most probable scale of the non-equilibrium Scale Distribution and any time t is given by,

$$\eta_{mp}(t) = \left(1 + \alpha e^{\frac{-t}{\tau}}\right)\eta$$

This is also written as,

$$\eta_{mp}(t) = \left(1 + \alpha e^{\frac{-t}{\tau}}\right)\left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$$

Let this equation be called the, "Scale Displacement Equation". So, the Scale Displacement Equation describes the shifting of the most probable length scale with respect to time, controlled by the time constant τ and α that depends upon the experimental setup and the fluid itself which defines the most probable scale. But, it is important to note that it describes the variation of most probable length scale only till the attainment of the Kolmogorov scale and only if external energy source does not supply energy after inducing turbulence. Let us analyze the variation of most probable length scale with time by plotting some graphs.

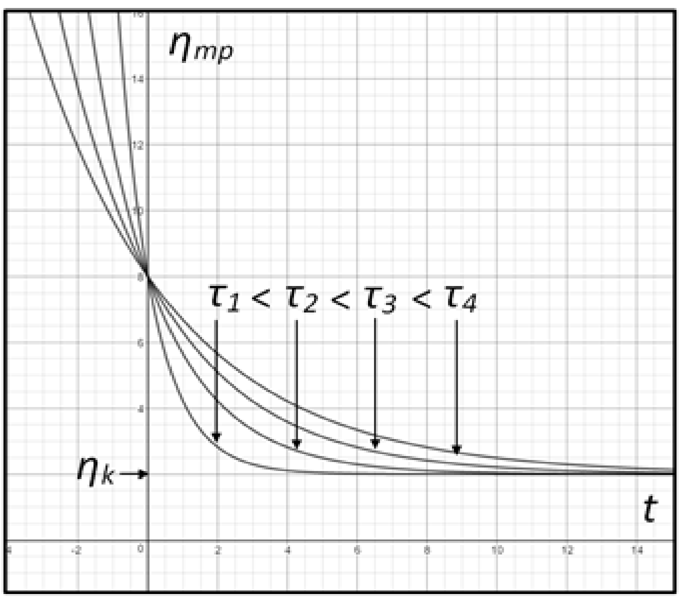

Fig. 1: *The variation of most probable length scale with time at different time constants keeping α constant. Time t is depicted on x-axis and η_{mp} on the y-axis. (graph not to scale).*

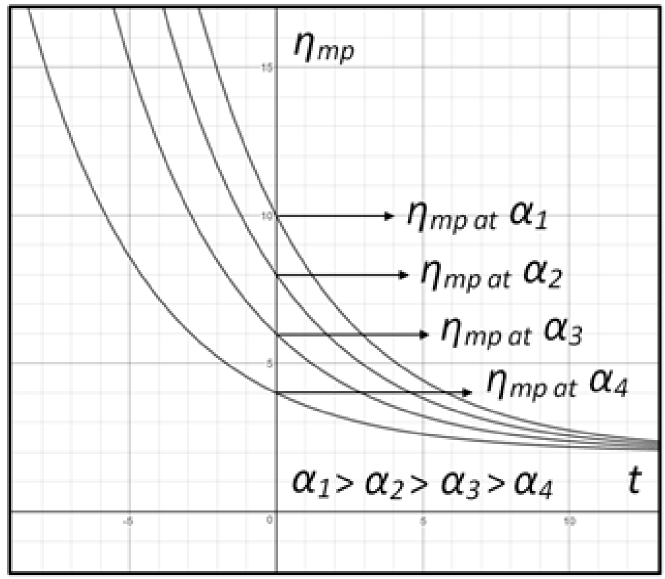

Fig. 2: *The variation of most probable length scales with time at different α's keeping τ constant. Time t is depicted on x-axis and η_{mp} on the y-axis. (graph not to scale). The most probable length scale is $\alpha\eta_k$, which is $\eta_{mp}(0)$; it decays to lower sizes and approaches the Kolmogorov Microscale with time. This leads to different y-intercepts.*

4 THE SPIN DISTRIBUTION

The Scale Distribution handles the variation of the scale of the eddies with respect to time, but, an essential feature of the eddies is their vorticity. Mainly, the direction of their spin and their angular velocities are some parameters that define the properties of the chaotic structure of turbulent flows. It has been observed that the direction of the spin of the eddies may be same as the parent eddy or may be opposite. Let us consider, the sign convention, that clockwise rotation is positive spin and counter-clockwise rotation is negative spin from observer's point of view. The magnitude of spin is defined by the angular velocity of any stationary point on the vortex in the frame of reference of that vortex. It can be postulated that, the statistical distribution that may describe the spins of all the eddies of a given system, is an even function. Suppose, f is an even function, then,

$$f(x) = f(-x)$$

Thus, the Probability Density Function for the Spin Distribution is also an even function. This is derived from the fact that it is equally likely to find eddies of same size and angular velocity to be of opposite spin in the system. It gives an important symmetry to the system; the distribution is symmetric about the y-axis; (fraction of total eddies having a particular spin); plotted against 'spin' on the x-axis. Now, according to Kolmogorov Theory, the energy cascade takes place and leads to an overall decrease in per capita spin energy of the eddies. So, as the eddies split, ideally half of the energy is inherited by the two daughter eddies. In similarity to the variation of most probable scale with time, the most probable spin, either positive or negative, approaches zero as scale approaches the Kolmogorov length scale. There's another aspect of the Spin Distribution, that it has to account for the interaction between the eddies, unlike the Scale Distribution. This interaction can be constructive or destructive. This implies if, two eddies of opposite spin and equal angular velocities meet, they cancel each other. Similarly, if the two eddies have same spin and angular velocity, they interact constructively. So, these properties of interference are fundamental to the isotropic view of turbulence. To harness interference effects in turbulent flows, we need to introduce some parameters that account for the same. These parameters are the Construction and Destruction Coefficients. Let, the Construction Coefficient be denoted by κ_c and the Destruction Coefficient be denoted by κ_d. These are numbers having dimensions of $[L^{-3}]$, and are defined as follows;

$$\kappa_c = \frac{n_c}{V} \quad \text{and} \quad \kappa_d = \frac{n_d}{V}$$

Where n_c and n_d are number densities of construction and destruction events in the system and V is the characteristic volume of the system. There is something fundamental to the nature of these events, namely, Construction and Destruction. Whenever these events occur, they are independent of each other and random due to chaos. These events occur at an ideally constant rate in isotropic turbulence. Such behaviour can be modelled by the Poisson Distribution. But, the Poisson distribution is a discrete probability distribution and construction events occur continuously at all scales. So there is a necessity to modify the Poisson distribution function. Poisson distribution function is given by,

$$P(x,\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

Where, λ is the number of expected occurrences of an event. In the context of turbulence, λ is replaced by some function of κ_c and κ_d, which will stand for number of constructive and destructive events occurring in unit volume, in the system. By similarity to the Scale Distribution, let, the most probable spin be denoted by, σ_{mp} and let it be (β+1) times the spin at Kolmogorov length scale. Now, spin at any scale η is given by,

$$\sigma_\eta = (\epsilon \cdot \eta)^{\frac{1}{3}}$$

Thus, at Kolmogorov length scale, spin is given by,

$$\sigma_{\eta_k} = (\epsilon \cdot \eta_k)^{\frac{1}{3}} = \left(\epsilon \cdot \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}\right)^{\frac{1}{3}}$$

Hence, the most probable spin in the system is given by,

$$\sigma_{mp} = (1+\beta)(\epsilon \cdot \eta_k)^{\frac{1}{3}}$$

Let us introduce the required modifications to the Poisson Distribution function, in the next section, which will account for continuous variations. As discussed earlier, the Poisson Distribution is a discrete probability distribution; we cannot evaluate the function at non-integers and negative numbers. So, to account for continuous variations of the variable time t (which will be used in the place of x) we need to use the Gamma Function, which is a continuous version of the Factorial function. The Gamma Function is defined by,

$$\Gamma(x) = \int_0^\infty t^{x-1}\, e^{-t}\, dt$$

While using the Poisson formula, in context of turbulence, we do not use the distribution function for computation of any probability, but for modelling time evolution of the most probable spin in the system; So, replacing λ by $(\kappa_c - \kappa_d)$•V and x by t/τ, where τ is the time constant for eddy construction or destruction, we get,

$$\sigma_{mp}(t) = \sigma_{mp} \cdot \underbrace{\frac{\left((\kappa_c - \kappa_d)V\right)^{\frac{t}{\tau}} \cdot e^{-(\kappa_d - \kappa_c)V}}{\Gamma\left(\frac{t}{\tau}\right)}}$$

Heuristically derived from Poisson Distribution Function.

This is the equation that describes the time evolution of the most probable spin in the system and allows us to find the most probable spin at any time t. It is important to note that, the exponential has the argument, $(\kappa_d - \kappa_c)$•V, Without this change, increasing κ_c resulted in decrease in most probable spin and increasing κ_d resulted in increase in most probable spin with respect to time, which is contrary to what is observed. So, the negation was a crucial modification of the equation. Now, to express this variation in terms of the Kolmogorov length scale, we use the relations on the previous page, and get the equation,

$$\sigma_{mp}(t) = (1+\beta)(\epsilon \cdot \nu)^{\frac{1}{4}} \cdot \frac{\left((\kappa_c - \kappa_d)V\right)^{\frac{t}{\tau}} \cdot e^{-(\kappa_d - \kappa_c)V}}{\Gamma\left(\frac{t}{\tau}\right)}$$

Where,
β controls the most probable spin in terms of Kolmogorov length scale.
ϵ, as discussed earlier is the rate of energy dissipation,
ν is kinematic viscosity of the fluid,
κ_c is the Construction Coefficient,
κ_d is the Destruction Coefficient,
V is the characteristic volume of the system,
Γ is the Gamma function,
t is time and,
τ is the time constant for construction or destruction.

Now, abiding to the sign convention for spin and considering the fact that the Probability density function for the Spin Distribution is an even function, the final equation may be given by,

$$\sigma_{mp}(t) = \pm\,(1+\beta)(\epsilon \cdot \nu)^{\frac{1}{4}} \cdot \frac{\left((\kappa_c - \kappa_d)V\right)^{\frac{t}{\tau}} \cdot e^{-(\kappa_d - \kappa_c)V}}{\Gamma\left(\frac{t}{\tau}\right)}$$

Let this equation be called the, 'Spin Interference Equation', which controls the most probable spin in the system parametrizing, the construction and destruction effects. Let us analyse the equations thus derived in the next section.

5 ANALYSIS

The derived equations, logically express the behaviour of a turbulent system, on statistical grounds. It interesting to know, that the Scale Displacement Equation, is a solution to a second order differential equation. This is due to the presence of two variable constants, α and τ. The differential equation for the Scale Displacement Equation is as follows,

$$\frac{d^2\eta_{mp}}{dt^2} - \frac{1}{(\eta_{mp} - \eta)}\left(\frac{d\eta_{mp}}{dt}\right)^2 = 0$$

Where, η_{mp} is the most probable scale, and η is the Kolmogorov length scale. It is an intuitive way to visualize the Scale Displacement Equation. One noticeable feature of the most probable scale is that, it is calculated as (α+1) times the Kolmogorov length scale. This is due to an important characteristic of the Scale Distribution, which is, there are two intervals of length that are of interest. These are, $[0,\eta_k]$ and $(\eta_k,\infty]$ on the scale axis. The most probable length scale is calculated with respect to η_k which is $\alpha\eta_k$ but with respect to origin, the most probable length scale is; $\alpha\eta_k + \eta_k$, which is same as writing, $\eta_{mp} = (\alpha+1)\eta_k$. The Spin Interference Equation is a relation which possesses the information about the variation of the most probable spin, either positive or negative, with respect to time. Depending upon the number density of construction and destruction events, the most probable spin either increases or decreases with time. If the number of construction events per unit volume is greater than the destruction counterpart, the most probable spin increases, but, only till a certain time, and then it decays. This happens due to the deficit of energy (which is required to sustain turbulence). This lack of inertial forces results in energy dissipation. Due to this characteristic, the most probable spin approaches the Kolmogorov spin, i.e. the spin at Kolmogorov length scale. Let us analyze the plots of the Spin Interference Equation, varying different quantities.

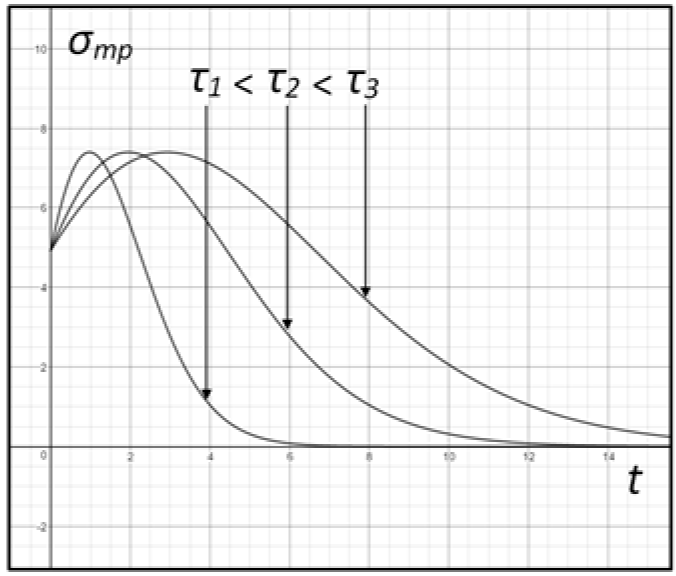

Fig. 3: *This graph depicts the variation of most probable spin (positive only), with time at different time constants for construction or destruction. As per the definition of the time constant, it is characterized by the average time interval between successive construction or destruction events. So, greater the time constant, slower is the variation of most probable spin with respect to time. Note the y-intercept, it is $\sigma_{mp}(0)$, which is the most probable spin at time t=0.*

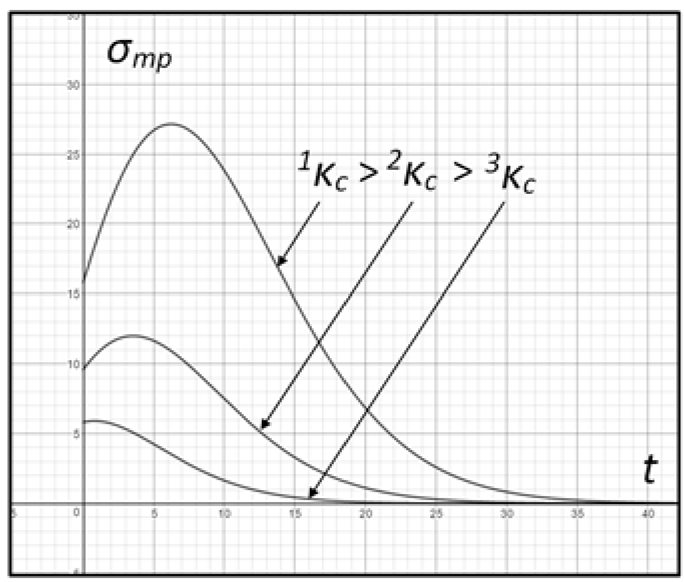

Fig. 4: *This plot describes the behaviour of the most probable spin at different Construction Coefficients. As these coefficients increase, the number of construction events per unit volume increase and leads to an increase in the most probable spin in the system. Eventually, every plot decays due to energy deficit. It is also important to note that, the peak of the graph, where it's derivative is zero, shifts forward in time, due to rise in the construction coefficient.*

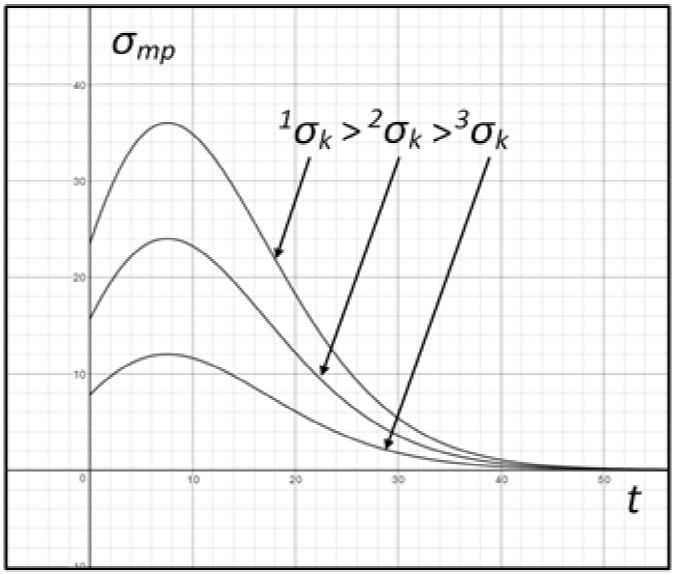

Fig. 5: *This graph depicts the variation most probable spin at different Kolmogorov spins.*

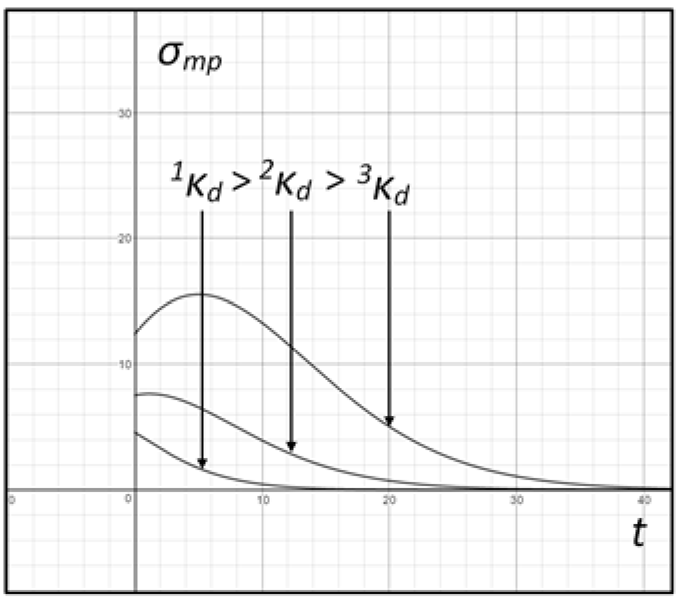

Fig. 6: *Similar to the plot for various construction coefficients, this graph depicts the variation of most probable spin on the basis of different destruction coefficients. Greater the destruction coefficient, quicker is the decay of most probable spin (which does not even increase when $\kappa_c \ll \kappa_d$).*

6 CONCLUSION

Turbulence is chaotic, random and intricate, which makes it an extremely difficult problem to harness using a mathematical model, particularly to describe its internal structures. Thus to describe various properties, macroscopic or microscopic, of a turbulent system, a statistical approach is a requisite. The Scale Displacement Equation and the Spin Interference Equation, thus derived, provide deep insight in understanding the properties of isotropic turbulence. They encode precise information about the variation of the most probable scale and spin of a turbulent system with time, parametrizing various factors. These beautiful equations inherit there properties naturally from Statistics and postulates of the Kolmogorov Theory of 1941. They treat static probability distributions as there foundation, and step into the regime of Non-Equilibrium Statistics, of time varying distributions. They also embody the unique use of the Poisson Distribution. Thus, the complex phenomenon of turbulence can be comprehended using coherent and precise equations, which explain the disordered internal self-interaction of a turbulent fluid. Hence, we improve our understanding of a fundamentally difficult problem. Hope for a better tomorrow!

7 REFERENCES

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