Approximate Solution For Time-Space Fractional Soil Moisture Diffusion Equation And Its Application

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Abstract: The purpose of this paper is to develop an implicit finite difference method for time-space fractional soil moisture diffusion equation (TSFSMDE). We prove a detailed analysis of the scheme and generate the discrete model. Also, we prove the scheme is unconditionally stable and convergent. As an application of the scheme we solve some test problems and their solutions are represented graphically by powerful software Mathematica.

IndexTerms: Time-Space fractional, soil moisture diffusion equation, finite difference, fractional derivatives, stability analysis, convergence analysis, Mathematica.

1 INTRODUCTION

In the present scenario fractional calculus plays an important role in the various fields of scientific and engineering problems. Fractional calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators. In the past few years, the increase of the subject is witnessed by series of conferences, research papers and several monographs [3], [4], [6], [7], [8], [9], [10], [11], [13], [14], [17], [19], [20], [21], [22], [23]. The dynamic models of a large number of phenomena can be modeled by fractional order partial differential equations (FOPDEs) which are characterized by fractional space and/or time derivatives. Fractional calculus is applied to model frequency dependent damping behavior of many viscoelastic materials, continuum and statistical mechanics, economics etc. Fractional diffusion equations have been used in modeling turbulent flow, chaotic dynamics of classical conservative system, groundwater contaminant transport, and applications in biology, physics, chemistry, finance etc. Schneider and Wyss [24] considered the time fractional diffusion and wave equations. As a matter of fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [16], [18]. But, most of the fractional differential equations do not have exact analytical solution, hence, approximation and numerical techniques must be used. In the last decade, extensive research has been carried out on the development of numerical methods for fractional partial differential equations. For the numerical solution of the fractional diffusion equations (FDE), many proposed methods have been initiated such as transform methods (Mainardi, 1997; Chaves, 1998; Agrawal, 2002), finite elements together with the method of lines (Liu, et al., 2004,

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El-Kahlout, 2008), explicit and implicit finite difference methods (Liu, et al., 2006; Meerschaert, et al., 2004; Shen, et al., 2005, Diego, 2008; Sweilam, et al., 2012) etc. Fractional diffusion equations have been topic of interest, studied by many researchers for developing fractional order finite difference schemes. Therefore, in context to this we develop the time-space fractional implicit finite difference scheme for fractional soil moisture diffusion equation. The solution of the linear partial differential equation of flow was first proposed by Casagrande, through the use of the graphical flowent method [2], [5], [12], [15]. This method is based on the assumptions that water flows region must be defined in terms of head or non-head flow. The flowent solutions proposed by Casagrande were for simple unconfined flow cases without flux boundary conditions. First experimental study on the movement of water in the soil was done by Henry Darcy (1856). Edgar Buckinghm (1907) described the water flow in unsaturated porous media modifying the equation of Darcy. Richard's (1931) combined the equations of Darcy and Buckingham with the equation of continuity to establish an overall relationship. Klute (1972) described several methods for estimating the hydraulic conductivity and diffusivity for unsaturated soils [1], [2], [15]. To understand such phenomenon, soil scientists have made some models for the flow of water into soil. Furthermore, many researchers developed different types of equations that model the water flow into soil. We consider the general diffusion equation of unsaturated flow of soil moisture as follows

$$\frac{\partial}{\partial x}(D\frac{\partial U}{\partial x})+\frac{\partial}{\partial y}(D\frac{\partial U}{\partial y})+\frac{\partial}{\partial z}(D\frac{\partial U}{\partial z})+\frac{\partial K}{\partial z}=\frac{\partial U}{\partial t} \qquad (1.1)$$

Where U(x, y, z, t) = the volumetric soil moisture content, D = the diffusivity of soil moisture, K = the capillary or hydraulic conductivity of soil moisture. If for (1.1), the flow takes place in the Z direction, as for infiltration of water into the soil, then (1.1) becomes one-dimensional flow equation, which is given below

$$\frac{\partial}{\partial z}(D\frac{\partial U}{\partial z})+\frac{\partial K}{\partial z}=\frac{\partial U}{\partial t}$$

Where D = K $\partial h_t / \partial U$, h_t = the tension head and K = the capillary conductivity. If the flow is considered in x direction (taken horizontal) then (1.1) becomes

$$\frac{\partial}{\partial x}(D\frac{\partial U}{\partial x})=\frac{\partial U}{\partial t} \quad (1.3)$$

Now we assume that D is a constant then the one-dimensional diffusion equation is

$$\frac{\partial U}{\partial t}=D\frac{\partial^2 U}{\partial x^2} \quad (1.4)$$

Which is exactly the diffusion heat flow equation [5] and it is well studied by Richard's [2] for water flow instead of heat flow. The model problem for the moisture flow in horizontal tube is given by

$$\frac{\partial U}{\partial t}=D\frac{\partial^2 U}{\partial x^2}, \ t > 0, \ x \geq 0 \quad (1.5)$$

To solve a particular model problem of moisture flow into a horizontal tube, we have to impose proper initial and boundary conditions. So, we consider an initial uniform moisture percentage of U as U_0 (U_0 is constant) at time t = 0, which becomes the initial condition and is mathematically expressed as

$$U(x, t) = U_0, \ t = 0, \ x \geq 0 \quad (1.6)$$

Now for left boundary condition, there is source of water applied and placed at x = 0 so as to maintain at all times after t=0 as U_L, and which is mathematically expressed as

$$U(x, t) = U_L, \ x = 0, \ t \geq 0 \quad (1.7)$$

Now for right boundary condition, there is source of water applied and placed at semi-infinite plane so as to maintain at all times after t = 0 is U_R, which is mathematically expressed as

$$U_x(x, t) = U_R, \ x \to \infty, \ t \geq 0 \quad (1.8)$$

Therefore, the model initial boundary value problem (IBVP) for soil moisture flow is given as follows

$$\frac{\partial U}{\partial t}=D\frac{\partial^2 U}{\partial x^2}, \ t > 0, \ x \geq 0 \quad (1.9)$$

Subject to the initial and boundary conditions

$$U(x, t) = U_0, \ t = 0, \ x \geq 0 \quad (1.10)$$

$$U(x, t) = U_L, \ x = 0, \ t \geq 0, \ U_x(x, t) = U_R, \ x \to \infty, \ t \geq 0 \quad (1.11)$$

for U(x, t) is volumetric water content and D is the diffusivity constant of soil moisture. We consider the following definitions of fractional derivatives which are useful for further developments

Definition 1.1 The Caputo time fractional derivative of order α, (0 < α ≤ 1) is defined as follows [16]

$$\frac{\partial^{\alpha}U(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial U(x,t)}{\partial \xi}\frac{\partial^2\xi}{(t-\xi)^{\alpha}}, & 0<\alpha<1\\ \frac{\partial U(x,t)}{\partial t}, & \alpha=1\end{cases}$$

Definition 1.2 The Grunwald-Letnikov space fractional derivative of order β, (1 <β ≤ 2) is defined as

$$\frac{\partial^{\beta}U(x,t)}{\partial x^{\beta}}=\frac{1}{\Gamma(-\beta)}\lim_{N\to\infty}\frac{1}{h^{\beta}}\sum_{j=0}^{N}\frac{\Gamma(j-\beta)}{\Gamma(j+1))}U[x-(j-1)h,t]$$

where Γ(.) is the gamma function. We organize the paper as follows: In section 2, we develop the implicit fractional order finite difference scheme for time-space fractional soil moisture diffusion equation. The stability of the solution is proved in section 3 and the concept of convergence is discussed in section 4. Numerical solution of time-space fractional soil moisture diffusion equation is obtained using Mathematica software in the last section.

2 FINITE DIFFERENCE SCHEME

We consider the following time-space fractional soil moisture diffusion equation (TSFSMDE) with initial and boundary conditions

$$\frac{\partial^{\alpha}U(x,t)}{\partial t^{\alpha}}=D\frac{\partial^{\beta}U(x,t)}{\partial x^{\beta}};(x,t)\in\Omega:[0,L]*[0,T] \quad (2.1)$$

initial condition: $U(x, 0) = U_0, \ 0 \leq x \leq L$ (2.2)

boundary conditions : $U(0, t) = U_L$,
$U_x(L,t)=0, x\to\infty, t\geq 0$ (2.3)

Where 0 < α ≤ 1; 1 < β ≤ 2 and D is the diffusivity constant. For the implicit numerical approximation scheme, we define $h=[(x_R - x_L)/N] = L/N$ and $\tau = T/N$, the space and time steps respectively, such that $t_k = k\tau$; k = 0,1,...,N be the integration time $0 \leq t_k \leq T$ and $x_i = x_L + ih$ for i = 0,1, ..., N. Define $U_i^k = U(x_i, t_k)$ and let U_i^k denote the numerical approximation to the exact solution $U(x_i, t_k)$. In the differential equation (2.1), the time fractional derivative term is approximated by the following scheme

$$\frac{\partial^{\alpha}U(x_i,t_{k+1})}{\partial t^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_{k+1}}\frac{1}{(t_{k+1}-\xi)^{\alpha}}\frac{\partial U(x_i,\xi)}{\partial\xi}d\xi$$

$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{j+1})-U(x_i,t_j)}{\Delta t}\int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^{\alpha}}$$

$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{j+1})-U(x_i,t_j)}{\tau}\int_{(k-j)\tau}^{(k+1-j)\tau}\frac{d\eta}{\eta^{\alpha}}$$

$$=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{U(x_i,t_{k+1-j})-U(x_i,t_{k-j})}{\tau}\int_{(j)\tau}^{(j+1)\tau}\frac{d\eta}{\eta^{\alpha}}$$

$$= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{U(x_{i,}t_{k+1-j})-U(x_{i,}t_{k-j})}{\tau}[(j+1)^{1-\alpha}-j^{1-\alpha}]$$

$$\therefore \frac{\partial^\alpha U(x_i,t_{k+1})}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U(x_{i,}t_{k+1})-U(x_{i,}t_k)]+ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}[U(x_{i,}t_{k+1-j})-U(x_{i,}t_{k-j})]b_j$$

Where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, j = 1,2…,k.

We discretise the spatial β-order fractional derivative using the Gr¨unwald finite difference formula at all time levels. The standard Gr¨unwald estimate generally yields unstable finite difference equation regardless of whatever result in finite difference method is an explicit or an implicit system for related discussion [6], [16], [18]. Therefore, we use a right shifted Gr¨unwald formula to estimate the spatial β-order fractional derivative

$$\frac{\partial^\beta U(x,t)}{\partial x^\beta} = \frac{1}{\Gamma(-\beta)}\lim_{N\to\infty}\frac{1}{h^\beta}\sum_{j=0}^{N}\frac{\Gamma(j-\beta)}{\Gamma(j+1)}U[x-(j-1)h,t]$$

where N is the positive integer, $h=[(x_R - x_L)/N]$ and Γ(.) is the gamma function. For the implicit numerical approximation scheme, we define $t_k = k\tau$ be the integration time $0 \le \tau \le T$ and $\Delta x = h > 0$ to be the grid size in x-direction, $h= [(x_R - x_L)/N]$ with $x_i = x_L + ih$ for i = 0,1, ..., N. Define $U_i^k = U(x_i, t_k)$ and let U_i^k denote the numerical approximation to the exact solution $U(x_i, t_k)$. We also define the normalized Gr¨unwald weights by $g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)}$, j = 0,1,2… For $D_t^\alpha U(x_i, t_{k+1})$, we adopt Caputo time fractional derivative for approximating the first order time derivative and for $D_x^\beta U(x_i, t_{k+1})$, we adopt the right shifted Gr¨unwald formula at all time levels for approximating the second order space derivative. Using time-space fractional approximation, the implicit type numerical approximation to (2.1) - (2.3) is given as follows

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U(x_{i,}t_{k+1})-U(x_{i,}t_k)]+ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j[U(x_{i,}t_{k+1-j})-U(x_{i,}t_{k-j})] = D\delta_{\beta,x}U_i^{k+1}$$

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U_i^{k+1}-U_i^k]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j[U_i^{k-j+1}-U_i^{k-j}] = D\,\delta_{\beta,x}\,U_i^{k+1} \quad (2.4)$$

Where the above fractional partial differential operator is defined as

$$\delta_{\beta,x}U_i^k = \frac{1}{h^\beta}\sum_{j=0}^{i+1}g_{\beta,j}U_{i-j+1}^k$$

Which is an $O(h^\beta)$ approximation to the β-order fractional derivative and $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, j = 1,2…,k. Therefore, the fractional approximated equation is

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}[U_i^{k+1}-U_i^k]+\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=1}^{k}b_j[U_i^{k-j+1}-U_i^{k-j}] = \frac{D}{h^\beta}\sum_{j=0}^{i+1}g_{\beta,j}U_{i-j+1}^{k+1} \quad (2.5)$$

After simplification, we get

$$[U_i^{k+1}-U_i^k]+\sum_{j=1}^{k}b_j[U_i^{k-j+1}-U_i^{k-j}] = r\sum_{j=0}^{i+1}g_{\beta,j}U_{i-j+1}^{k+1} \quad (2.6)$$

where

$$r = \frac{D\tau^\alpha\Gamma(2-\alpha)}{h^\beta},\ g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)} \text{ and}$$

$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, j = 1,2,…k.

The initial condition is approximated as $U_i^0=U_0$, i = 1, 2, …, N. The left boundary condition is approximated as $U_0^k=U_L$, k=0,1,2,…,N. Now using central difference the right boundary condition is approximated as follows

$$\frac{U_{N+1}^k - U_{N-1}^k}{2h} = 0, k = 0,1,2.......N$$

Therefore, the fractional approximated IBVP is

$$(1+r\beta)U_i^1 - r\sum_{j=0, j\neq 1}^{i+1}g_{\beta,j}U_{i-j+1}^1 = U_i^0, \text{ for k = 0} \quad (2.7)$$

$$(1+r\beta)U_i^{k+1} - r\sum_{j=0, j\neq 1}^{i+1}g_{\beta,j}U_{i-j+1}^{k+1} = (1-b_1)U_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U_i^{k-j} + b_kU_i^0, \text{ for k} \ge 1 \quad (2.8)$$

initial condition : $U_i^0 = U_0$, 1, 2, ...,N. (2.9)

boundary conditions: $U_o^k = U_L$ and

$U_{N+1}^k = U_{N-1}^k$ (2.10)

where,

$$r = \frac{D\tau^\alpha\Gamma(2-\alpha)}{h^\beta},\ g_{\beta,j} = \frac{\Gamma(j-\beta)}{\Gamma(-\beta)\Gamma(j+1)} \text{ and}$$

$b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, j = 1,2,…,k.

Therefore, the fractional approximated IBVP (2.7) − (2.10) can be written in the following matrix equation form:

$AU^1 = U^0 + B$, for k = 0 (2.11)

$$AU^{k+1} = (1-b_1)U^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})U^{k-j} + b_k U^0 + B$$

for k ≥ 1 (2.12)

Where $U^k = (U_1^k, U_2^k, ..., U_N^k)$, k = 0, 1, 2...,N and

$$\begin{pmatrix} 1+r\beta & -rg_{\beta}.0 & 0... & & & 0 \\ -rg_{\beta}.2 & 1+r\beta & -rg_{\beta}.0 & 0... & & 0 \\ -rg_{\beta}.3 & -rg3.2 & 1+r\beta & 0... & & 0 \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots \\ \vdots & \vdots\ddots & \ddots & \vdots\ddots & & \vdots \\ \text{-rg}_{\beta,}\text{N-1} & \text{-rg}_{\beta,}\text{N-2} & \cdots & & 1+r\beta & \text{-rg}_{\beta,}0 \\ \text{-rg}_{\beta,}\text{N} & \text{--rg}_{\beta,}\text{N-1} & \cdots & & \text{-r(-rg}_{\beta,o}\text{+g}_{\beta,2}) & 1+r\beta \end{pmatrix}$$

That is, A = (aij) is a N^{th} order square matrix of coefficients. For i = 0, 1, 2, ...,N, j = 0, 1, 2, ...,N, the coefficients are

$$a_{ij} = \begin{cases} 0, \text{ when } j \ge i+2 \\ -rg_{\beta,0}, \text{ when } j = i+1 \\ 1 + r\beta, \text{ when } j = i = 1,2,4......, \\ -rg_{\beta,j}, \text{ otherwise } j = 2,3,4 ...., N \end{cases} \quad (2.13)$$

While $a_{N,N-1} = -r(g_{\beta,0} + g_{\beta,2})$ and

$B = (rg_{\beta,2} U_L, rg_{\beta,3} U_L, \cdots, rg_{\beta,N} U_L, rg_{\beta,N+1} U_L)^T$.

The above system of algebraic equations is solved by using Mathematica software in section 5. In the next section, we discuss the stability of the solution of time space fractional implicit finite difference scheme (2.7) − (2.10) for the time space fractional soil moisture diffusion equation (TSFSMDE) (2.1) − (2.3).

3 STABILITY

Theorem 3.1 The solution of the discretised scheme (2.7) − (2.10) for the time space fractional soil moisture diffusion equation (2.1) − (2.3) is unconditionally stable.

Proof: We have for i = 1, 2, · · ·, N; k = 1, 2, · · · , N, the coefficients $g_{\beta,j}$, j = 1, 2, ...,N satisfy the following equations

(i) $g_{\beta,0} = 1$; $g_{\beta,1} = -\beta < 0$; $g_{\beta,j} > 0$ for j ≠ 1;

(ii) $\sum_{j=0}^{\infty} g_{\beta,j} = 0$ and $\sum_{j=1}^{i+1} g_{\beta,j} \prec 0$

We assume that $\overline{U_i^k}$ is a vector of exact solution of TSFSMDE (2.1) − (2.3). Therefore, $E_i^k = \overline{U_i^k} - U_i^k$ for i=0,1,...N; k=0,1, ...N satisfy the following equations

$$AU^1 = U^0 + B$$

$$AU^{k+1} = C_1U^k + C_2U^{k-1} + C_3U^{k-2} + ... + C_kU^1 + b_kU^0 + B$$

That is

$$AE^1 = E^0 + B$$

$$AE^{k+1} = C_1E^k + C_2E^{k-1} + C_3E^{k-2} + ... + C_k E^1 + b_k E^0 + B$$

where $E^0 = 0$ and $E^k = (\epsilon_1^k, \epsilon_2^k, ..., \epsilon^k_{N-1})^T$ and B includes a column vector of known boundary values and known source term values. Furthermore, we assume that

$\left|E_l^k\right| = \max_{1\le i\le N-1}\left|\epsilon_i^k\right| = \left\|E^k\right\|_\infty$, for l = 1,2,…

Therefore, from (2.7), we get

$$\left|E_l^1\right| = \left|(1+r\beta)\epsilon_i^1 - r\sum_{j=0}^{i+1} g_{\beta,j}\,\epsilon_{i-j+1}^1\right|$$

$\le |\epsilon_i^0| \le |E_l^0| = \|E^0\|_\infty$

Suppose that $\left\|E^k\right\|_\infty \le \left\|E^0\right\|_\infty$.

From (2.8), we get

$$\left|E_l^{k+1}\right| = \left|(1+r\beta)\epsilon_i^{k+1} - r\sum_{j=0,j\ne1}^{i+1} g_{\beta,j}\,\epsilon_{i-j+1}^{k+1}\right|$$

$$\le \left|(1-b_1)\epsilon_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})\epsilon_i^{k-j} + b_k\,\epsilon_i^0\right|$$

$$\le (1-b_1)\left|\epsilon_l^k\right| + (b_1 - b_k)\left|\epsilon_l^{k-j}\right| + b_k\left|\epsilon_l^0\right|$$

$$\le (1-b_1+b_1-b_k+b_k)\left|E_l^k\right|$$

$$\le \left|E_l^k\right| = \left\|E^k\right\|_\infty \le \left\|E^0\right\|_\infty$$

That is, $\left\|E^{k+1}\right\|_\infty \le \left\|E^0\right\|_\infty$.

Hence, by induction we prove the scheme is unconditionally stable. The next section is devoted for convergence of the finite difference scheme.

4 CONVERGENCE

Theorem 4.1 The fractional order implicit finite difference scheme (2.7) − (2.10) for TSFSMDE (2.1) − (2.3) is unconditionally convergent. Proof: Let us assume that $\left|e_l^k\right| = \max_{1\le i\le N-1}\left|\epsilon_i^k\right| = \left\|E^k\right\|_\infty$, for i = 1,2,… and $T_i^k = \max_{1\le i\le N}\left|T_i^k\right|$.

Using $\sum_{j=0}^{\infty} g_{\beta,j} = 0$ and $\sum_{j=1}^{i+1} g_{\beta,j} < 0$, then from (2.7) we get,

$$(1+r\beta)\left|e_l^1\right| = \left|(1+r\beta)e_i^1 - r\sum_{j=0,\, j\neq 1}^{i+1} g_{\beta,j}e_{i-j+1}^1\right|$$

$\leq |e_i^0| + |T_i^1|$, where $T_i^k = h^2[O(\tau + h^2)]$

$$\Rightarrow \left|e_l^1\right| \leq \frac{1}{(1+r\beta)}\left|e_i^0\right| + \frac{1}{(1+r\beta)}\left|T_i^1\right|$$

$\leq |e_l^0| + c\,|T_i^1|$, (for $c = 1/(1+r\beta)$

$\leq |e_l^0| + c\,h^2[O(\tau + h^2)]$

$$\Rightarrow \left\|E^1\right\|_\infty \leq \left\|E^0\right\|_\infty + cO(\tau + h^2)$$

Suppose that

$$\left\|E^k\right\|_\infty \leq \left\|E^0\right\|_\infty + cO(\tau + h^2)$$

From (2.8), we get

$$(1+r\beta)\left|E_l^{k+1}\right| = \left|(1+r\beta)e_i^{k+1} - r\sum_{j=0,\, j\neq 1}^{i+1} g_{\beta,j}e_{i-j+1}^{k+1}\right|$$

$$\leq \left|(1-b_1)e_i^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})e_i^{k-j} + b_k e_i^0\right| + \left|T_i^k\right|$$

$$\leq (1-b_1)\left|e_l^k\right| + (b_1 - b_k)\left|e_l^{k-j}\right| + b_k\left|e_l^0\right| + \left|T_i^k\right|$$

$$\leq (1-b_1+b_1-b_k+b_k)\left|E_l^{k-j}\right| + \left|T_l^k\right|$$

$$\Rightarrow \left|E_l^{k+1}\right| \leq \left\|E_l^k\right\| + \frac{1}{(1+r\beta)}\left|T_l^k\right|$$

$\leq |E_l^k| + cO(\tau + h^2)$

$$\Rightarrow \left\|E^{k+1}\right\|_\infty \leq \left\|E^0\right\|_\infty + cO(\tau + h^2).$$

Therefore, from this we observe that for any x and t as h and τ approach to zero in such way that $(ih,k\tau)\to(x,t)$, $U_i^k \to U(x, t)$ as $(h, \tau) \to (0, 0)$. So, this proves that U_i^k converges to $U(x_i, t_k)$ as $(h, \tau) \to (0, 0)$. Hence, the proof of the theorem is completed.

5 NUMERICAL SOLUTIONS

In this section, we obtain the approximated solution of time-space fractional soil moisture diffusion equation with initial and boundary conditions. To obtain the numerical solution of the time-space fractional soil moisture diffusion equation (TSFSMDE) by the finite difference scheme, it is important to use some analytical model. Therefore, we present an example to demonstrate that TSFSMDE can be applied to simulate behavior of a fractional diffusion equation by using Mathematica software. We consider the following, dimensionless one-dimensional time-space fractional soil moisture diffusion equation with suitable initial and boundary conditions

$$\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} = \frac{\partial^\beta U(x,t)}{\partial x^\beta},\ 0<x<1,\ 0<\alpha\leq 1,\ 1<\beta\leq 2,\ t>0$$

initial condition : $U(x, 0) = 0$, $0 \leq x \leq 1$

boundary conditions : $U(0, t) = 1$, $U_x(x, t) = 0$, as $x\to\infty$, $t > 0$, with the diffusion coefficient $D = 1$. The numerical solutions are obtained at $t = 0.05$ by considering the parameters $\tau = 0.005$, $h = 0.1$, simulated in the following figure.

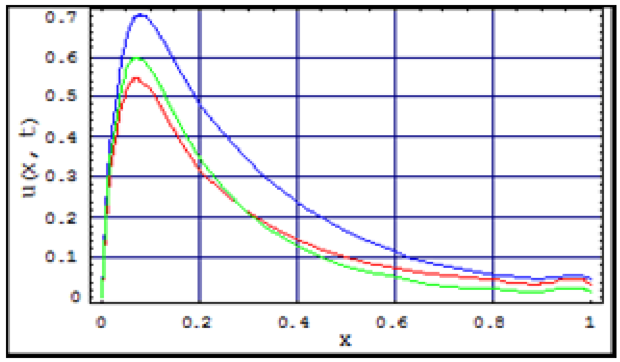

*Fig.*5.1 : The soil moisture diffusion profile with $t = 0.05$, $h = 0.1$, $\alpha = 0.7$, $\beta = 1.7$(*red*), $\alpha = 0.8$, $\beta = 1.9$(*blue*) *and* $\alpha = 0.9$, $\beta = 1.8$(*green*).

CONCLUSIONS

(i) We successfully developed the fractional order implicit finite difference scheme for time space fractional soil moisture diffusion equation in a bounded domain.
(ii) Analysis of the scheme shows that the numerical results are in good agreement with our theoretical analysis.
(iii) We observe that the fractional order implicit finite difference scheme is numerically stable.

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