Method For Solving Fuzzy Integro-Differential Equation By Using Fuzzy Laplace Transformation

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Abstract: In this article we study the method of solving fuzzy integro-differential equation under certain condition by using fuzzy Laplace transformation . Finally we give some illustrative examples.

Keywords: Fuzzy numbers, Volterra integro-differential equation, fuzzy Laplace Transformation, Rimann-integrable , Hausdorff distance .

1. Introduction

The concept of fuzzy sets and set operations was first introduced by Zadeh [19] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Abbasbandy et. al [12] introduced a numerical algorithm for solving linear Fredholm fuzzy integral equations of second kind by using parametric form of fuzzy number and converting a linear fuzzy Fredholm integral equations to two linear systems of integral equation of the second kind in crisp case Badolian et. al studied another numerical method for solving linear fuzzy Fredholm integral equation of second kind by using Adomian method. Moreover , Friedman et. al [8] investigated an embendding method to solve fuzzy Voltera and Fredholm integral equations. However many other authors obtained the numerical integration of fuzzy valued functions and solving fuzzy Voltera and Fredholm equations. The concept of fuzzy Laplace Transformation was introduced by Allahviranloo , Ahmadi [16] . After many other researcher use it to solve fuzzy differential equations, fuzzy integral equations etc. Some authors discussed the solution of fuzzy integro-differential equation by fuzzy differential transform method in their research paper [13] .

2. Definition and Background

A fuzzy number is a fuzzy subset of the real line *R* i.e a function $u: R \to [0,1]$ which is bounded, convex and normal. Let *E* denote the set of all fuzzy numbers which are upper semi continuous and have compact support. The α- level set $[u]_\alpha$ set of a fuzzy real number *u* for $0 < \alpha \le 1$, defined as $[u]_\alpha = \{ t \in R : u(t) \ge \alpha\}$. A fuzzy real number *u* is called *convex*, if $u(t) \ge u(s) \wedge u(r) = \min(u(s), u(r))$, where $s < t < r$. If there exists $t_0 \in R$ such that $u(t_0) = 1$, then the fuzzy real number *u* is called *normal*. A fuzzy real number *u* is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $u^{-1}([0, a + \varepsilon))$, for all $a \in [0,1]$ is open in the usual topology of *R*.

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The absolute value $|u|$ of $u \in E$ is defined as (see for instance Kaleva and Seikkala

$$|u|(t) = \max\{u(t), u(-t)\}, \quad \text{if } t \ge 0$$
$$= 0, \quad \text{if } t < 0.$$

It is clear that that the α- level set of the fuzzy number u is a closed and bounded interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$ where $\underline{u}(\alpha)$ denotes the left-hand end point and $\overline{u}(\alpha)$ denotes the right-hand end point of $[u]_\alpha$. Two arbitrary fuzzy numbers $\bar{u} = [\underline{u}(\alpha), \overline{u}(\alpha)]$ and $\bar{v} = [\underline{v}(\alpha), \overline{v}(\alpha)]$ are said to be equal i.e $\bar{u} = \bar{v}$ if and only if $\underline{u}(\alpha) = \underline{v}(\alpha)$ and $\overline{u}(\alpha) = \overline{v}(\alpha)$. Since each $y \in R$ can be regarded as a fuzzy number $\bar{y}$ defined by :

$$\bar{y}(t) = \begin{cases} 1 & if \quad t = y \\ 0 & if \quad t \neq y \end{cases}$$

The Hausdorff distance between fuzzy numbers is a mapping $\bar{d}: L(R) \times L(R) \to R_+$ defined by

$$d(u,v) = \sup_{0 \le \alpha \le 1} \max\left\{\left|\underline{u}(\alpha) - \underline{v}(\alpha)\right|, \left|\bar{u}(\alpha) - \bar{v}(\alpha)\right|\right\}$$

Where $u = [\underline{u}(\alpha), \overline{u}(\alpha)]$ and $= [\underline{v}(\alpha), \overline{v}(\alpha)]$. It can be easily shown that d is a metric on $L(R)$ with the following properties :

1. $d(u+w, v+w) = d(u,v)$ for all $u, v, w \in L(R)$.
2. $d(ku, kv) = |k| d(u,v)$ for all $u, v \in L(R)$
3. $d(u+v, w+e) \le d(u,w) + d(v,e)$ for all $u, v, w, e \in L(R)$
4. $(d, L(R))$ is a complete metric space.

Definition2.1: Let $f: R \to L(R)$ be a fuzzy valued function. f is said to be continuous at $t_0 \in R$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(f(t), f(t_0)) < \varepsilon \quad \text{whenever} \quad |t - t_0| < \delta .$$

Definition2.2: Let $f: R \to L(R)$ be a fuzzy valued function and $x_0 \in R$. We say that f is differentiable at x_0, if there exists $f'(x_0) \in L(R)$ such that

$$(a)\lim_{h\to 0+}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0+}\frac{f(x_0)-f(x_0-h)}{h}$$
$$= f'(x_0)$$

Or

$$(b)\lim_{h\to 0-}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0-}\frac{f(x_0)-f(x_0-h)}{h}$$
$$= f'(x_0)$$

Theorem 2.1: Let $f: R \to L(R)$ be a fuzzy valued function and denote $f(t)=\left[\underline{f}(t,\alpha),\overline{f}(t,\alpha)\right]$ for each $0\le\alpha\le 1$ followings are hold –

- **(a)** If f is differentiable in the first form (a) in definition 2.2 , then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable and $f'(t)=\left[\underline{f}'(t,\alpha),\overline{f}'(t,\alpha)\right]$.
- **(b)** If f is differentiable in the 2nd form (b) in definition 2.2 , then $\underline{f}(t,\alpha)$ and $\overline{f}(t,\alpha)$ are differentiable and $f'(t)=\left[\overline{f}'(t,\alpha),\underline{f}'(t,\alpha)\right]$.

Theorem 2.2: Let $f: R \to L(R)$ be a fuzzy valued function and denote $f(t)=\left[\underline{f}(t,\alpha),\overline{f}(t,\alpha)\right]$ for each $0\le\alpha\le 1$, Then

- **(a)** If f and f' are differentiable in the first form (a) in definition 2.2 or if f and f' are differentiable in the 2nd form (b) in definition 2.2, then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t)=\left[\underline{f}''(t,\alpha),\overline{f}''(t,\alpha)\right]$.
- **(b)** If f is differentiable in the first form (a) and f' is differentiable in the 2nd form (b) or if f is differentiable in the 2nd form (b) and f' is differentiable in the first form (a) in definition 2.2, then then $\underline{f}'(t,\alpha)$ and $\overline{f}'(t,\alpha)$ are differentiable and $f''(t)=\left[\overline{f}''(t,\alpha),\underline{f}''(t,\alpha)\right]$.

Theorem 2.3: Let $f(x)$ be a fuzzy real valued function on $[a,\infty)$ and it is represented by $\left[\underline{f}(x,\alpha),\overline{f}(x,\alpha)\right]$. For any fixed $r\in[0,1]$, assume $\underline{f}(x,\alpha),\overline{f}(x,\alpha)$ are Rimann-integrable on $[a,b]$ for every $b\ge a$ and let there exists two positive $\underline{M}(\alpha)$ and $\overline{M}(\alpha)$ such that $\int_a^b\left|\underline{f}(x,\alpha)\right|dx\le\underline{M}(\alpha)$ and $\int_a^b\left|\overline{f}(x,\alpha)\right|dx\le\overline{M}(\alpha)$ for every $b\ge a$. Then $f(x)$ is improper fuzzy Rimann-integrable on $[a,\infty)$ and the improper fuzzy Rimann-integral is a fuzzy number. Further more , we have

$$\int_a^\infty f(x)dx=\left(\int_a^\infty\underline{f}(x,\alpha)dx,\int_a^\infty\overline{f}(x,\alpha)dx\right)$$

Proposition2.1: Let $f(x)$ and $g(x)$ be a fuzzy real valued function and also fuzzy Rimann-integrable on $I=[a,\infty)$,then $f(x)+g(x)$ is Rimann-integrable on $I=[a,\infty)$ and,

$$\int_I[f(x)+g(x)]dx=\int_I f(x)dx+\int_I g(x)dx$$

Definition 2.3: Fuzzy Laplace Transformation

The fuzzy Laplace transform of a fuzzy real valued function $f(t)$ is defined as follows:

$$F(s)=L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\lim_{T\to\infty}\int_0^T e^{-st}f(t)dt$$

whenever the limit exist.

The symbol L is fuzzy Laplace transformation, which acts on fuzzy real valued function $f=f(t)$ and generates $F(s)=L\{f(t)\}$. The lower and upper Laplace transform of a fuzzy real valued function $f(t)$ are given as follows:

$$F(s,\alpha)=L\{f(t,\alpha)\}=\left[l\{\underline{f}(t,\alpha)\},l\{\overline{f}(t,\alpha)\}\right]$$

Where

$$l\{\underline{f}(t,\alpha)\}=\int_0^\infty e^{-st}\underline{f}(t,\alpha)dt=\lim_{T\to\infty}\int_0^T e^{-st}\underline{f}(t,\alpha)dt$$

, $0\le\alpha\le 1$

$$l\left\{\overline{f}(t,\alpha)\right\}=\int_0^\infty e^{-st}\,\overline{f}(t,\alpha)dt=\lim_{T\to\infty}\int_0^T e^{-st}\,\overline{f}(t,\alpha)dt$$

, $0\le\alpha\le1$

Definition 2.3: Fuzzy Convolution Theorem :
The convolution of two fuzzy real valued functions f,g defined for $t\ge0$ by

$$(f*g)(t)=\int_0^t f(T)g(t-T)dT$$

Theorem 2.4 : If f and g are piecewise continuous fuzzy real valued function on $[0,\infty)$ with exponential order p, then-

$$L\{(f*g)(t)\}=L\{f(t)\}.L\{g(t)\}=F(s).G(s)\quad,\quad s>p\ .$$

3. Main Method

Consider the following Fuzzy Volterra Integro-Differential equation :

(1)... $u^{(n)}(t)=f(t)+\int_0^t k(t-s)u(s)ds$ with

$$u^{(k)}(0)=\overline{a}=\left(\underline{a_k},\overline{a_k}\right);0\le k\le n-1.$$

Where $u^{(n)}$ denotes the n^{th} derivative of u.

Taking Fuzzy Laplace transformation on both sides of (1) , we get

$$L[u^{(n)}(t)]=L[f(t)]+L[\int_0^t k(s-t)u(s)]ds$$

By definition of Fuzzy Laplace transformation and using Fuzzy Convolution Theorem ,

we have ,

$$s^nL\left[u(t)\right]-s^{n-1}u(0)-s^{n-2}u'(0)-.......u^{(n-1)}(0)=$$
$$L[f(t)]+L[\int_0^t k(s-t)u(s)]ds$$

This implies that

(2).
$$s^n l\left\{\underline{u}(t;\alpha)\right\}-s^{n-1}\underline{a_0}-s^{n-2}\underline{a_1}......\underline{a_{n-1}}=l\left\{\underline{f}(t;\alpha)\right\}+\underline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}\ ,$$
$$0\le\alpha\le1$$

and

(3)...
$$s^n l\left\{\overline{u}(t;\alpha)\right\}-s^{n-1}\overline{a_0}-s^{n-2}\overline{a_1}......\overline{a_{n-1}}=l\left\{\overline{f}(t;\alpha)\right\}+\overline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}$$
$$,\ 0\le\alpha\le1$$

Now we discuss the following cases :

Case 1: If $u(t;\alpha)$ and $k(t;\alpha)$ are both positive , then

$$\underline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\underline{k(t;\alpha)}\right\}l\left\{\underline{u(t;\alpha)}\right\}$$

$$\overline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\overline{k(t;\alpha)}\right\}l\left\{\overline{u(t;\alpha)}\right\}$$

Case 2: If $u(t;\alpha)$ is negative and $k(t;\alpha)$ is positive , then

$$\underline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\overline{k(t;\alpha)}\right\}l\left\{\underline{u(t;\alpha)}\right\}$$

$$\overline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\underline{k(t;\alpha)}\right\}l\left\{\overline{u(t;\alpha)}\right\}$$

Case 3: If $u(t;\alpha)$ is positive and $k(t;\alpha)$ is negative , then

$$\underline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\underline{k(t;\alpha)}\right\}l\left\{\overline{u(t;\alpha)}\right\}$$

$$\overline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\overline{k(t;\alpha)}\right\}l\left\{\underline{u(t;\alpha)}\right\}$$

Case 4: If $u(t;\alpha)$ and $k(t;\alpha)$ are both negative , then

$$\underline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\overline{k(t;\alpha)}\right\}l\left\{\overline{u(t;\alpha)}\right\}$$

$$\overline{l\left\{k(t;\alpha)\right\}l\left\{u(t;\alpha)\right\}}=l\left\{\underline{k(t;\alpha)}\right\}l\left\{\underline{u(t;\alpha)}\right\}$$

We introduce explicit formula for **Case1** and others are similar. So we have from (2) and (3)

$$s^n l\left\{\underline{u}(t;\alpha)\right\} - s^{n-1}\underline{a_0} - s^{n-2}\underline{a_1}\ldots\ldots\underline{a_{n-1}}$$
$$= l\left\{\underline{f}(t;\alpha)\right\} + l\left\{\underline{k(t;\alpha)}\right\} l\left\{\underline{u(t;\alpha)}\right\}, \quad 0 \le \alpha \le 1$$

and

$$s^n l\left\{\overline{u}(t;\alpha)\right\} - s^{n-1}\overline{a_0} - s^{n-2}\overline{a_1}\ldots\ldots\overline{a_{n-1}}$$
$$= l\left\{\overline{f}(t;\alpha)\right\} + l\left\{\overline{k(t;\alpha)}\right\} l\left\{\overline{u(t;\alpha)}\right\}$$

In compact form:

$$l\left\{\underline{u}(t;\alpha)\right\} = \frac{l\left\{\underline{f}(t;\alpha)\right\} + s^{n-1}\underline{a_0} + s^{n-2}\underline{a_1}.+\ldots..+\underline{a_{n-1}}}{\left(s^n - l\left\{\underline{k(t;\alpha)}\right\}\right)}, \quad 0 \le \alpha \le 1$$

and

$$l\left\{\overline{u}(t;\alpha)\right\} = \frac{l\left\{\overline{f}(t;\alpha)\right\} + s^{n-1}\overline{a_0} + s^{n-2}\overline{a_1}.+\ldots..+\overline{a_{n-1}}}{\left(s^n - l\left\{\overline{k(t;\alpha)}\right\}\right)}, \quad 0 \le \alpha \le 1$$

By taking the inverse of fuzzy Laplace transformation on both sides of above relations we can easily obtain the value of $\underline{u}(t;\alpha)$ and $\overline{u}(t;\alpha)$ where $0 \le \alpha \le 1$.

Example 3.1: To solve the following Fuzzy Volterra Integro-Differential equation

$$u''(t) = (\alpha + 2, 4 - \alpha)t + \int_0^t (t - x)u(x)\,dx \quad ;$$
$$u(0) = (\alpha + 1, 3 - \alpha); u'(0) = (\alpha, 2 - \alpha)$$

By using fuzzy Laplace transformation on both sides of the equation , we get

$$L[u''(t)] = L[(\alpha + 2, 4 - \alpha)t] + L[t]L[u(t)]$$

This implies

$$s^2 L\left\{u(t)\right\} - su(0) - u'(0) = L\{(\alpha + 2, 4 - \alpha)t\}$$
$$+ L\{t\}L\{u(t)\}$$

i.e

$$s^2 l\left\{\underline{u}(t;\alpha)\right\} = l\left\{(\alpha+2)t\right\} + l\left\{t\right\} l\left\{\underline{u}(t;\alpha)\right\}$$
$$+ s(\alpha+1) + (\alpha),$$

and

$$s^2 l\left\{\overline{u}(t;\alpha)\right\} = l\left\{(4-\alpha)t\right\} + l\left\{t\right\} l\left\{\overline{u}(t;\alpha)\right\}$$
$$+ s(3-\alpha) + (2-\alpha),$$
$$0 \le \alpha \le 1$$

i.e

$$l\left\{\underline{u}(t;\alpha)\right\} = (\alpha+2)\left(\frac{1}{s^4 - 1}\right) + (\alpha+1)\left(\frac{s^3}{s^4 - 1}\right)$$
$$+ (\alpha)\left(\frac{s^2}{s^4 - 1}\right),$$
$$0 \le \alpha \le 1$$

and

$$l\left\{\overline{u}(t;\alpha)\right\} = (4-\alpha)\left(\frac{1}{s^4 - 1}\right) + (3-\alpha)\left(\frac{s^3}{s^4 - 1}\right)$$
$$+ (2-\alpha)\left(\frac{s^2}{s^4 - 1}\right),$$
$$0 \le \alpha \le 1$$

By taking the inverse of fuzzy Laplace transformation on both sides of above relations, we have

$$\underline{u}(t;\alpha) = (\alpha+2).\frac{1}{2}(\sinh t - \sin t) +$$
$$(\alpha+1).\frac{1}{2}(\cos t + \cosh t) + (\alpha)\frac{1}{2}(\sin t + \sinh t)$$
$$\overline{u}(t;\alpha) = (4-\alpha).\frac{1}{2}(\sinh t - \sin t) + (3-\alpha).$$
$$\frac{1}{2}(\cos t + \cosh t) + (2-\alpha)\frac{1}{2}(\sin t + \sinh t),$$

$$0 \le \alpha \le 1.$$

Example 3.2: Consider the Fuzzy Volterra Integro-Differential equation

$$u'''(t) = (\alpha+1, 3-\alpha)e^t + \int_0^t \sin(t-x)u(x)\,dx$$
$$u(0) = u'(0) = u''(0) = \bar{0} = (0,0)$$

By using fuzzy Laplace transformation on both sides of the equation , we get

$$l\{\underline{u}(t;\alpha)\} = (\alpha+1)\left(\frac{s^2+1}{(s^5+s^3-1)(s-1)}\right) \quad , \quad 0 \le \alpha \le 1$$

$$l\{\bar{u}(t;\alpha)\} = (3-\alpha)\left(\frac{s^2+1}{(s^5+s^3-1)(s-1)}\right) \quad , \quad 0 \le \alpha \le 1$$

By taking the inverse of fuzzy Laplace transformation on both sides of above relations we can easily obtain the value of $\underline{u}(t;\alpha)$ and $\bar{u}(t;\alpha)$ where $0 \le \alpha \le 1$.

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