On The Ring Structure Of Soft Set Theory

Singh, D., Onyeozili, I.A.

Abstract: - Molodtsov in 1999 initiated the concept of soft set theory. Based on the work of Molodtsov, Maji et al. in 2003 introduced some of the operations of soft sets and gave some of their basic properties. Ali et al. in 2009 further introduced more new operations on soft sets. The algebraic structures of soft sets were studied by Aktas and Cagman who introduced the notions of soft groups and their algebraic properties, while Acar et al studied soft semi rings and their basic properties. In this paper, we discuss the notions of soft rings, soft sub rings, soft ideal of a soft ring and idealistic soft rings and some of their algebraic properties with some illustrative examples.

Index terms: - Soft sets, Soft rings, Soft sub rings, Soft ideal, Idealistic Soft ring, Soft ring homophomism.

---

1. INTRODUCTION

More often than not, the type of problems encountered in real life are inherently uncertain, imprecise and complicated and so cannot be successfully solved using classical methods and other existing mathematical theories such as probability Theory, Fuzzy Set Theory[13], Rough Set Theory[12], Vague Set Theory[7] etc., due to their inherent difficulties. The major reason for these difficulties according to Molodtsov[11] is possibly due to the inadequacies of their parameterization tools. So to overcome these difficulties, Molodtsov[11] introduced the concept of Soft Set as a completely new mathematical tool for dealing with these uncertainties. Recently, soft set theory has been developed rapidly and focused by some scholars in theory and practice. Based on the work of Molodtsov[11],Maji et al[10] and Ali et al[3] published detailed theoretical studies on operations of Soft Sets and their algebraic properties. Soft set theory has also a rich potential for applications in many directions, some of which have been discussed in [11]. Maji et al[9] and Cogman and Enginoglu[5] applied soft set theory to decision making problems. Aktas and Cogman[2] studied the basic concept of soft groups and their properties, while U. Acar et al.[1] introduced soft semi rings and discussed their basic properties. Since then some researchers A.Segzin and A.O Atagun[4] , Y.B Jun[8] and Celik et al[6] have studied other soft algebraic structures as well as their properties. The rest of this paper is organized as follows;

- **a.** Section 2 gives some known and useful preliminary definitions and notations on soft set theory and ring theory.
- **b.** Section 3 discusses soft rings and idealistic soft rings, while Section 4 concludes the paper.

- Professor D. Singh (Former Professor, Indian Institute of Technology, Bombay) Mathematics Department Ahmadu Bello University Zaria, Nigeria
  Email: mathdss@yahoo.com Tel: (+234) 8022704813
- Onyeozili, I.A. (corresponding author) Department of Mathematics, University of Abuja, Nigeria (PhD Research Scholar) Email: ijeozili@gmail.com
  Tel: (+234) 8033149663

2. PRELIMINARIES

2.1 Soft Set Theory

In this subsection, we give some known and useful definitions and notations on Soft Set Theory. In what follows, let *U* be an initial universe set and *E* a set of parameters with respect to *U*. Let *P(U)* denote the power set of *U* and *A* be a subset of *E*.

Definition 2.1 [11]

A pair *(F,A)* is called a *soft set* over *U*, where *F* is a mapping $F:A\to P(U)$. In other words, a soft set over *U* is a parameterized family of subsets of the universe *U*. For $x \in A$, *F(x)* may be considered as the set of *x-approximate* elements of the soft set *(F,A)* i.e $(F,A) = \{ F(x) \in P(U) : x \in A \subseteq E \}$

Definition 2.2 [1]

For a soft set *(F,A)*, the set supp(*F,A*), called the *support* of *(F,A)* is given by Supp(*F,A*) = $\{ x \in A \mid f(x) \neq \emptyset\}$. If Supp *(F,A)* ≠ ∅, then the soft set *(F,A)* is called non-null.

Definition 2.3 [11]

Let *(F,A)* and *(G,B)* be two soft sets over *U*. Then:

- ***i.*** *(F,A)* is said to be a *soft subset* of *(G,B)* denoted by $(F,A) \subseteq (G,B)$, if $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$.
- **ii.** *(F,A)* and *(G,B)* are said to be *soft equal* denoted by *(F,A) = (G,B)* if $(F,A) \subseteq (G,B)$ and $(G,B) \subseteq (F,A)$.

Definition 2.4 [3]

- **i.** The *restricted intersection* of *(F,A)* and *(G,B)* over a common universe *U*, denoted by $(F,A) \Cap (G,B)$ is the soft set *(H,C)* where $C=A\cap B$ and for all $x \in C$.

$$H(x) = F(x) \cap G(x).$$

- ***ii.*** The *union* of *(F,A)* and *(G,B)* over a common universe *U*, denoted by $(F,A) \tilde{\cup} (G,B)$ is the soft set *(H,C)*, where $C=A\cup B$ and for all $x \in C$,

$$H(x)=\begin{cases} F(x), \text{if } x \in A - B; \\ G(x), \text{if } x \in B - A \text{and} \\ F(x)\cup G(x), \text{ if } x\in A\cap B. \end{cases}$$

Definition 2.5 [10]

Let *(F,A)* and *(G,B)* be two soft sets over *U*. Then:

The *AND-product* or AND-intersection of *(F,A) and (G,B)* denoted by $(F,A) \wedge (G,B)$ is the soft set *(H,C),* where $C = A \times B$ and for all $(x,y) \in A \times B$, $H(x,y) = F(x) \cap G(y)$.

Definition 2.6[3]
Let $\{(H_i, C_i)$ i∊I, the index set} be a family of soft sets over *U*. Then;

i. The restricted intersection of the family (H_i, C_i) denoted by $⋒_{i\epsilon I}(H_i, C_i)$, is the soft set

(H,C) where $C = \cap_{i\epsilon I} C_i$ and for all $x \in C$,

$$H(x) = \cap_{i\epsilon I} H_i(x).$$

ii. The union of the family (H_i,C_i) denoted by $\tilde{\cup}_{i\epsilon I}(H_i, C_i)$, is the soft set (H,C) where

$C = U_{i\epsilon I} C_i$ and for all $x \in C$.

$$H(x) = \cup_{i\epsilon I}(x)\ Hi(x).$$

iii. The ∧ - product of the family (H_i, C_i) denoted by $\wedge_{i\epsilon I}(H_i, C_i)$ is the soft set (H,C) where

$C = \prod_{i\epsilon I} C_i$, and for all $(x_i)_{i\epsilon I} \in C$.

$$H((x_i)_{i\epsilon I}) = \cap_{i\epsilon I} H_i(x_i).$$

Definition 2.7 [3]
Let *(F,A)* be a soft set over *U.* Then;

i. *(F,A)* is said to be a relative null soft set, denoted by $\mathcal{N}_A$, if $f(x)=\emptyset$ for all $x \epsilon A$.

ii. *(F,A)* is said to be a relative whole soft set, denoted by $\mathcal{W}_A$, if $f(x)=U$, for all $x \epsilon A$.

2.2 Basic Definitions On Ring Theory
Some useful definitions and notations of the theory of rings are given in this subsection.

Definition 2.8 [6]
A *ring* $(R,+,.)$ is an algebraic structure consisting of a non-empty set *R* together with two binary operations on *R*, called addition(+) and multiplication(.) such that;

i. $(R,+)$ is a commutative group.

ii. $(R,.)$ is a semi group an**d**

iii. $a(a+b) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

If *R* contains an element *I* such that $Ia = aI$ for all $a \in R$, then *R* is said to be a *ring with identity*. A zero element of a ring *R* is an element *o*(necessarily unique) such that $o+a=a+o=a$ for all $a \epsilon R$. A non-empty subset *S* of a ring *R* is called a *subring* if and only if $a\text{-}b \in S$ and $ab \in S$ for all $a, b \in S$.A non-empty subset *I* of *R* is called *an ideal*, denoted by $I \lhd R$ is and only if $a - b \in I$ and $ra, ar \in I$ for all $a, b \in I$ and $r \in R$. The subrings $\{o\}$ and *R* are called *trivial sub rings* of *R*.

Definition 2.9[1]
Let R_1 and R_2 be rings. A mapping $f{:}R_1 \rightarrow R_2$ is called a *ring homomorphism* if it satisfies;

$F(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b, \in R_1$. That is the mapping *f* preserves the ring operations.

A ring homomorphism $f{:}\ R_1 \rightarrow R_2$ is called a *monomorphism [resp. epimorphism, isomorphism*] if it is an injective (resp. surjective, bijective) mapping. The *kernel* of a ring homomorphism $f{:}R_1 \rightarrow R_2$ is its kernel as a map of additive groups, that is; $Ker\ f = \{\ r \in R_1 \mid f(r) = o\}$. Similarly the *image* of *f*, denoted by *Im.f* is the set $\{s \in R_2 \mid f(r) = s$ for some $r \in R_1\}$.

Theorem 2.1[6]
Let $\{R_i \mid i\epsilon I\}$ be a family of subrings(ideals) of *R.* Then their intersection $\cap_{i\epsilon I} R_i$ is a subring(ideal) of *R.*

Theorem 2.2[6]
Let $\{S_i \mid i\epsilon I\}$ be a family of subrings(ideals) of R_i Then $\prod_{i\epsilon I} Si$ is a subring(ideal) of $\prod_{i\epsilon I} R_i$

3. SOFT RINGS AND IDEALISTIC SOFT RINGS
In this section, we discuss the notions of soft rings, soft subrings, soft ideal of a soft ring, idealistic soft rings and soft ring homomorphisms. We also discuss some of their algebraic properties with some illustrative examples.

Definition 3.1 [1]
Let *R* be a ring and let *(F,A)* be a non-null soft set over *R*. Then *(F,A)* is called a *soft ring* over *R* if *F(x)* is a subring of *R* , denoted by $F(x) < R,\ \forall x \in A$.

Example 3.1
Let $R=A=\mathbb{Z}_6=\{0, 1, 2, 3, 4, 5\}$. Consider the soft set *(F,A)* over *R*, where $F{:}A \rightarrow P(R)$ is a set valued function defined by $F(x)=\{y \in R \mid x.y=0\}$. Then $F(_0) = R$, $F(_1) = \{0\}$, $F(_2) = \{0 , 3\}$, $F(_3) = \{0, 2, 4\}$, $F(_4) = \{0, 3\}$ and $F(_5) = \{0\}$ which are all subrings of *R* . Hence *(F,A)* is a soft ring over *R*.

Theorem 3.1[1]
Let *(F,A)* and *(G,B)* be soft rings over *R.* Then;

1. $(F,A) \wedge (G,B)$ is a soft ring over *R* , if it is non-null.
2. $(F,A) ⋒ (G,B)$ is a soft ring over *R* , if it is non-null.
3. $(F,A)\ \tilde{\cup}\ (G,B)$ is a soft ring over *R,* if $A \cap B = \emptyset$

Proof:

1. Let $(F,A) \wedge (G,B) = (H,C)$, where $C = A \times B$ and $H(x,y) = F(x) \cap G(y)$ for all $(x,y) \in C$.

If *(H,C)* is a non-null, then $H(x,y) = F(x) \cap G(y) \neq \emptyset$

Since *F(x)* and *G(y)* are subrings of *R,* then *H(x,y)* is a subring of *R* (by theorem2.1)

Therefore *(H,C)* is a soft ring over *R.*

2. Let $(F,A) ⋒ (G,B) = (H,C)$, where $C= A \cap B = \emptyset$, and $H(x) = F(x) \cap G(x)$ for all $x \in C$.

Also $H(x) = F(x) \cap G(x) \neq \emptyset$ for all $x \in Supp(H,C)$,since *F(x)* and *G(x)* are subrings of *R*

Then, $H(x) < R$

Therefore *(H,C)* is a soft ring over *R.*

3. Let $(F,A)\ \tilde{\cup}\ (G,B) = (H,C)$, where $C = A \cup B$, and;

$$H(x)=\begin{cases}F(x), x\in A-B;\\ G(x), x\in B-A\\ F(x)\cup G(x),\ x\in A\cap B.\end{cases}$$

Since $A \cap B = \emptyset$, $F(x) \cup G(x) = \emptyset$

Therefore for all $x \in C$

$$H(x)=\begin{cases}F(x), x\in A-B\\ G(x), x\in B-A\end{cases}$$

Also $H(x) < R$ since $F(x)$ and $G(x)$ are sub rings of R. Therefore (H,C) is a soft ring over R. Generalizing the above theorem, we have the following;

Theorem 3.2 [1]
Let $(F_i,A_i)_{i\epsilon I}$, where I is an index set, be a non empty family of soft rings over R. Then;

1. $\wedge_{i\epsilon I}(F_i,A_i)$ is a soft ring over R , if it is non-null.
2. $⋒_{i\epsilon I}(F_i,A_i)$ is a soft ring over R, if it is non-null.
3. $\tilde{\cup}_{i\epsilon I}(F_i,A_i)$ is a soft ring over R, if $A_i \cap A_j=\emptyset$, $i\neq j$, $i, j\epsilon I$

Proof

(1) Let $\wedge_{i\epsilon I}(F_i,A_i) =(H,C)$, where $C=\prod_{i\epsilon I}A_i$ and $H(x_i)=\cap_{i\epsilon I}F_i(x_i)$ for all $(x_i)_{i\epsilon I}\ \epsilon\ C$. If $H(C)$ is a non-null and $(x_i)_{i\epsilon I}\ \epsilon\ supp(H,C)$ then $H((x_i)_{i\epsilon I}) = \cap_{i\epsilon I} F_i(x_i)\neq\emptyset$ and $F_i(x_i) < R$, $\forall\ i\epsilon I$. Hence $H((x_i)_{i\epsilon I}) < R$ for all $(x_i)_{i\epsilon I}\ \epsilon\ supp(H,C)$. Therefore (H,C) is a soft ring over R.

(2) Let $⋒_{i\epsilon I}(F_i,A_i) =(H,C)$, where $C=\cap_{i\epsilon I}A_i$ and $H(x)=\cap_{i\epsilon I} F_i(x)$ for all $x\epsilon C$. If (H,C) is a non-null, and $x\ \epsilon\ supp(H,C)$, then $H(x)= \cap_{i\epsilon I} F_i(x)\neq\emptyset\Longrightarrow F_i(x)<R, \forall\ i\epsilon I\Longrightarrow H(x)<R, \forall\ x\ \epsilon\ supp\ (H,C)$. Therefore (H,C) is a soft ring over R.

(3) Let $\tilde{\cup}_{i\epsilon I}(F_1,A_1) = (H,C)$, where $C=\cup_{i\epsilon I}A_i$ and $\forall\ x\epsilon C$, $H(x)= \cup_{i\epsilon I}(x)F_i(x)$ where $I(x)=\{i\epsilon I\ \ x\epsilon A_i\}$. (H,C) is a non-null since $supp(H,C)= \cup_{i\epsilon I} supp(F_i,A_i)\neq\emptyset$. Therefore $H(x)$ $\cup_{i\epsilon I}(x)F_i(x)\neq\emptyset\Longrightarrow F_{io}(x)\neq\emptyset$, for some $i_o\epsilon I(x)$ since $A_i \cap A_j=\emptyset$, $i\neq j$, then i_o is unique. Therefore $H(x)=F_{io}(x)<R$. Therefore $H(x)<R$. Hence (H,C) is a soft ring over R.

Definition 3.2 [1]
Let (F,A) and (G,B) be soft rings over R. Then (G,B) is called a soft sub ring of (F,A), If it satisfies the following;

i. $B\subset A$
ii. $G(x)$ is a sub ring of $F(x)$, for all $x\ \epsilon\ supp(G,B)$.

Example 3.2
Let $R=A=2\mathbb{Z}$ and $B=6\mathbb{Z}\subset A$. Consider the set-valued functions $F:A\rightarrow P(R)$ and $G:B\rightarrow P(R)$ given by $F(x) = \{nx \mid n\epsilon\mathbb{Z}\}$ and $G(x) = \{5nx \mid n\epsilon\mathbb{Z}\}$. Clearly $G(x)=5x\mathbb{Z}$ is a sub ring of $F(x)=x\mathbb{Z}$, $\forall\ x\epsilon B$.

Hence (G,B) is a soft sub ring over (F,A).

Theorem 3.3 [1]
Let (F,A) and (G,B) be soft rings over R. Then;

1. If $G(x) \subset F(x)$, $\forall\ x\epsilon B\subset A$, then (G,B) is a soft sub ring of (F,A).
2. $(F,A)⋒(G,B)$ is a soft sub ring of both (F,A) and (G,B) if it is non-null.

Proof

(1) Since $B\subset A$ and $G(x)\subset F(x) \vee x\epsilon\ B$, then $G(x) < F(x)\ \forall\ x\epsilon B$ Therefore (G,B) is a soft sub ring of (F,A).

(2) Let $(F,A)\ ⋒\ (G,B) = (H,C)$, where $C=A\cap B\subset A$ and $H(x)=F(x) \cap G(x) \neq\emptyset$ is a sub ring of $F(x)\ \forall\ x\ \epsilon\ supp\ (H,C)$. Hence (H, C) is a soft sub ring of (F, A).

Example 3.3
Let $R= \mathbb{Z}$, $A=2\mathbb{Z}$ and $B=3\mathbb{Z}$. Consider the set-valued functions $F:A\rightarrow P(R)$ and $G:B\rightarrow P(R)$ defined by $F(x)=\{2nx \mid n\epsilon\ \mathbb{Z}\} = 2x\mathbb{Z}$ and $G(x)=\{3nx\mid n\epsilon\mathbb{Z}\}$. Let $(F,A)\ ⋒\ (G,B) = (H,C)$, where $C=A\cap B=6\mathbb{Z}$ and $\forall\ x\epsilon C$, $H(x)=F(x)\cap G(x)=6x\mathbb{Z}$ which is a sub ring of both $F(x)$ and $G(x)$. Hence (H,C) is a soft sub ring of both (F,A) and (G,B).

Definition 3.3 [1]
Let (F,A) be a soft ring over R. A non-null soft set (I,B) over R is called *a soft ideal* of (F,A) denoted by $(I,B)\lhd (F,A)$, if it satisfies the following conditions:

i. $B\subset A$
ii. $I(x)$ is an ideal of $F(x)$ for all $x\ \epsilon\ supp(I,B)$.

Example 3.4
Let $R=A=\mathbb{Z}_4=\{0,1,2,3\}$ and $B=\{0,1,2\}$. Consider the set-valued function $F:A\rightarrow P(R)$, defined by $F(x)=\{y\epsilon R \mid x.y\ \epsilon\ \{0,2\}\}$.Then we have that $F(0)=R$, $F(_1)=\{0,2\}$, $F(_2)=\{0,1,2,4\}$ and $F(_3)=\{0,2\}$ which are all sub rings of R. Therefore (F,A) is a soft ring over R.

Consider the function $I:B\rightarrow P(R)$ given by $I(x)=\{y\epsilon R \mid x.y=0\}$. Then we have that $I(_0)=R\lhd F(_0)$, $I(_1)=\{0\}\lhd F(_1)=\{0,2\}$ and $I(_2)=\{0,2\}\lhd F(_2)= \mathbb{Z}_4$

Hence $(I,B) \lhd (F,A)$.

Theorem 3.4 [1]
Let (I_1,A_1) and (I_2,A_2) be soft ideals of a soft ring (F,A) over R. Then;

1. $(I_1,A_1)\ ⋒\ (I_2,A_2)$ is a soft ideal of (F,A) if it is non-null.
2. $A_1 \cap A_2 = \emptyset$, then $(I_1,A_1)\ \tilde{\cup}\ (I_2,A_2)$ is a soft ideal of (F,A).

Proof

(1) Let $(I_1,A_1)\ ⋒\ (I_2,A_2) = (I,B)$, where $B=A1\cap A_2\subset A$. But $I_1(x)\lhd F(x)$ and $I_2(x)\lhd F(x)$ for all $x\ \epsilon\ supp(I,B)$. Therefore $I(x)=I_1(x)\cap I_2(x)\neq\emptyset$ since (I,B) is non-null. $\Rightarrow I(x) \lhd F(x)$. Hence $(I,B)\lhd (F,A)$

(2) Let $(I_1,A_1)\ \tilde{\cup}\ (I_2,A_2) = (I,C)$ where $C=A_1\cup A_2$ and $\forall\ x\epsilon C\subset A$,

Therefore $I(x)=I_1(x)$ is a non-empty ideal of $F(x)\ \forall x\epsilon\ supp(I,C)$

Also $I(x) = I_2(x)$ is a non-empty ideal of $F(x)\ \forall x\ \epsilon\ supp(I,C)$.

Therefore $I(x)$ is a non-empty ideal of $F(x)\ \forall x\ \epsilon\ supp(I,C)$.

Hence *(I,C)* is a soft ideal of *(F,A)*

Theorem 3.5[1]
Let (I_1,A_1) and (I_2,A_2) be soft ideals of soft rings, *(F,A)* and *(G,B)* over *R* respectively. Then $(I_1,A_1) \Cap (I_2,A_2)$ is a soft ideal of $(F,A) \Cap (G,B)$, if it is non-null.

Proof
Let $(I_1,A_1) \Cap (I_2,A_2) = (I,D)$, where $D=A_1 \cap A_2$ and $\forall\, x \in D$. $I(x)=I_1(x) \cap I_2(x)$. Let $(F,A) \Cap (G,B)=(H,C)$ where $C=A\cap B$ and $\forall\, x \in C$, $H(x)=F(x)\cap G(x)$. Since *(I,D)* is non-null, $I(x)=I_1(x) \cap I_2(x) \neq \emptyset$. Since $A_1 \subset A$ and $A_2 \subset B \Longrightarrow A_1 \cap A_2 \subset A\cap B$ ie $D \subset C$. Also since $I_1(x) \lhd F(x)$ and $I_2(x) \lhd G(x)$ $\forall\, x \in D$. $I_1(x) \cap I_2(x) \subset F(x) \cap G(x)$ $\forall x \in D \Rightarrow I(x) \subseteq H(x) \Rightarrow I(x) < H(x)$. Finally we need to show that $a.r \in I(x)$ $\forall\, r \in H(x)$ and $\forall\, a \in I(x)$. Since $I_1(x) \lhd F(x)$, $r \in H(x)=F(x)\cap G(x)$ and $a \in I(x)=I_1(x) \cap I_2(x)$. We have that $a.r \in I_i(x)$ and $a.r \in I_2(x)$. Hence $a.r \in I(x)$. Therefore $(I,D) \lhd (H,C)$.

Example 3.5
Let $R=M_2(\mathbb{Z})$ ie 2×2 matrices with terms in integers, $A=3\mathbb{Z}$, $B=5\mathbb{Z}$, $A_1=6\mathbb{Z}$ and $A_2=10\mathbb{Z}$. Consider the set-valued functions $F:A \rightarrow P(R)$ and $G:B \rightarrow P(R)$, defined by;

$$F(x)=\left\{\begin{bmatrix} nx & 0 \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z} \right\} and \ G(x) \left\{\begin{bmatrix} nx & nx \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z} \right\}$$

which are subrings of *R.* Thus *(F,A)* and *(G,B)* are soft rings over *R*. Consider the set-valued function $I_1:A_1 \rightarrow P(R)$ and $I_2:A_2 \rightarrow P(R)$ defined by;

$$I_1(x) = \left\{\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z} \right\} and \ I_2(x)=\left\{\begin{bmatrix} 0 & nx \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z} \right\}$$

which are ideals of *F(x)* and *G(x)* respectively. For $x \in A_1 \cap A_2$

$$I_1(x) \cap I_2(x)=\left\{\begin{bmatrix} 0 & nx \\ 0 & 0 \end{bmatrix} \mid n \in \mathbb{Z} \right\} \ and$$

$$F(x) \cap G(x)=\left\{\begin{bmatrix} nx & 0 \\ 0 & nx \end{bmatrix} \mid n \in \mathbb{Z} \right\}$$

Therefore $I_1(x) \cap I_2(x) \lhd F(x) \cap G(x)$. Since $A_1=6\mathbb{Z} \subset 3\mathbb{Z} = A$ and $A_2=10\mathbb{Z} \subset 5\mathbb{Z} = B \Rightarrow A_1 \cap A_2 = 30\mathbb{Z} \subset 15\mathbb{Z} = A \cap B$. Therefore $(I_1,A_1) \Cap (I_2,A_2) \lhd (F,A) \Cap (G,B)$.

Theorem 3.6[1]
Let $(F_i,A_i)_{i\epsilon I}$ be a non-empty family of soft ideals of a soft ring *(F,A)* over *R.* Then;

(1) $\Cap_{i\epsilon I}(F_i,A_i)$ is a soft ideal of *(F,A),* if it is non-null.
(2) $\wedge_{i\epsilon I}(F_i,A_i)$ is a soft ideal of *(F,A),*if it is non-null.
(3) $\tilde{\cup}_{i\epsilon I}(F_i,A_i)$ is a soft ideal of *(F,A),* if $\{A_i \ \ i\epsilon I\}$ are pair wise disjoint, whenever it is non-null.

Proof
The proof follows the same arguments in the proof of theorem 3.1 using Definition 3.3.

Definition 3.4 [1]
Let *(F,A)* be a non-null soft set over *R.* Then *(F,A)* is called an idealistic soft ring over *R*. If *F(x)* is an ideal of *R,* for all $x \in supp(F,A)$.

Example 3.6
In Example (3.1) and (3.2), *(F,A)* are idealistic soft rings over *R,* since *F(x)* is an ideal of *R,* for all $x \in A$.

Proposition 3.1 [1]
Let *(F,A)* be an idealistic soft ring over *R*. If $B \subset A$, then *(F,B)* is also an idealistic soft ring over *R,* if it is non-null.

Proof
Since *(F,A)* is an idealistic soft ring over *R,* then *F(x)* is an ideal of *R,* for all $x \in supp(F,A)$. Since $B \subset A \Rightarrow F(x)$ is also an ideal of *R,* for all $x \in supp(F,B)$. Therefore *(F,B)* is an idealistic soft ring over *R.*

Remark
Note that every idealistic soft ring over *R* is a soft ring over *R* but the converse is not necessarily true.

Theorem 3.7 [1]
Let *(F,A)* and *(G,B)* be idealistic soft rings over *R.* Then

1. $(F,A) \Cap (G,B)$ is an idealistic soft ring over *R,* if it is non-null.
2. $(F,A) \wedge (G,B)$ is an idealistic soft ring over *R,* if it is non-null.
3. $(F,A) \tilde{\cup} (G,B)$ is an idealistic soft ring over *R,* if $A\cap B = \emptyset$.

Proof:
Similar to theorem (3.1) in view of Definition (3.4) and Remark (3.1).

Remark
In theorem(3.7), if *A* and *B* are not disjoint, then the result is not true in general because the union of two different ideals of a ring *R* may not be an ideal of *R* as shown in Example(3.7) below.

Example 3.7
Let $R= \mathbb{Z}_{10}=\{0,1,2,3,4,5,6,7,8,9\}$, $A=\{0,4\}$ and $B=\{4\}$. Consider the set-valued function $F:A \rightarrow P(R)$ given by $F(x)=\{y \epsilon R \mid x.y=0\}$. Then $F(_0)=R \lhd R$ and $F(_4)=\{0,5\} \lhd R$. Hence *(F,A)* is an idealist soft ring over *R.* Now consider the set-valued function $G:B \rightarrow P(R)$ given by $G(x)=\{0\} \cup \{y \epsilon R\} \mid x+y \in \{0,2,4,6,8\}\}$. Then $G(_4)=\{0,2,4,6,8\} \lhd R$. Therefore *(G,B)* is an idealistic soft ring over *R.* But $F(_4) \cup G(_4)=\{0,2,4,5,6,8\}$ is not an ideal over *R* since 5 – 2 =3 ∉ {0,2,4,5,6,8}. Therefore $(F,A) \cup (G,B)$ is not an idealistic soft ring over *R.*

Example 3.8
To illustrate theorem (3.7), let

$$R=\left\{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid \ x,y,z \in \mathbb{Z} \right\} A=6\mathbb{Z} \ \ and \ \ B=10\mathbb{Z}$$

Consider the function *F:A→P(R)* and *G:B→P(R)* defined by

$$F(x)=\left\{\begin{bmatrix} nx & nx \\ 0 & 0 \end{bmatrix} \mid n\in\Box \right\} \; and \; G(x)=\left\{\begin{bmatrix} 0 & nx \\ 0 & nx \end{bmatrix} \mid n\in\Box \right\}$$

Therefore *(F,A)* and *(G,B)* are idealistic soft rings over *R.* Let *(F,A) ∧(G,B)=(H,C),* where *C=A×B* and *∀(x,y) ϵ A×B.*

$$H(x,y) = F(x) \cap G(y)= \left\{\begin{bmatrix} 0 & tx \\ 0 & 0 \end{bmatrix} \mid n\in\Box \right\} \lhd R$$

Where *t* is equal to the least common multiple(L.C.M) of *x* and *y.* Hence *(H,C)* is an idealistic soft ring over *R.*

Definition 3.5 [1]

- **i.** An idealistic soft ring *(F,A)* over a ring *R* is said to be trivial if *F(x) ={0}* for every *x ϵ A.*
- ***ii.*** An idealistic soft ring *(F,A)* over *R* is said to be whole if *F(x) = R* for all *x ϵ R.*

Example 3.9

Let *P* be a prime integer, *R= ℤp* and *A= ℤp-{0}.* For example if *P = 2,* then *R=* $\mathbb{Z}_2$*={0,1}* and *A={1}.* If *P=3,* then *R=* $\mathbb{Z}_3$*={0,1,2}* and *A={1,2}.* Consider the set-valued function *F:A→P(R)* given by *F(x)={yϵR │ (x.y)*$^{p-1}$*=1} ∪{0}.* Then for all *xϵA F(x)=R ⊲ R,* Hence *(F,A)* is a whole idealistic soft ring over *R.* Now consider the function *G:A→P(R)* defined by *G(x)={yϵR│x.y=0}.* Then *∀xϵA G(x)={0} ⊲ R.* Hence *(G,A)* is a trivial idealistic soft ring over *R.*

Definition 3.6[1]

Let *(F,A)* be a soft set over *R* and let *f:R→*R^1 be a mapping of rings. Then we can define a soft set *(f(F)A)* over R^1 where *f(F):A→P(R)* is defined as *f(F)(x)=f(F(x))* for all *xϵA* and that *supp(f(F),A)=supp(F,A).*

Proposition 3.2[1]

Let *f:R→*R^1 be a ring epimorphism. If *(F,A)* is an idealistic soft ring over *R*, then *(f(F),A)* is an idealistic soft ring over *R.*

Proof

Since *(F,A)* is a non-null, by definition (   ) then (*f(F),A)* is a non-null soft set over R^1 and for all *x ϵ supp(f(F),A), f(F)(x)=f(F(x))=∅.* ,But *F(x) ⊲ R* and *f* is an epimorphism. It implies *f(F(x))* is an ideal of R^1 for all *x ϵ supp(f(F),A).* Hence *(f(F),A)* is an idealistic soft ring over R^1*.*

Definition 3.7 [6]

Let *(F,A)* and *(G,B)* be a soft ring over *R* and R^1 respectively. Let *f:R→*R^1 and *g:A→B* be two mappings. The pair *(f,g)* is called a soft ring homomorphism if the following conditions are satisfied;

- **i.** *F* is a ring epimorphism
- **ii.** *g* is surjective
- ***iii.*** *f(F(x)=G(g(x))* for all *xϵA.*

If we have a soft ring homomorphism between *(F,A)* and *(G,B)* ,*(F,A)* is said to be soft ring homomorphism to *(G,B),* denoted by *(F,A) ~ (G,B).* In addition, if *f* is a ring isomorphism and *g* is a bijective, then *(f,g)* is called a soft ring isomorphism and we say that *(F,A)* is softly isomorphic to *(G,B)* denoted by *(F,A)≃(G,B).*

Example 3.10

Let *R=ℤ* and *R*$'$*={0}×ℤ, A=2ℤ, and B={0}×6ℤ.* Consider the set-valued functions *F:A→P(R)* and *G:B→P(R)* defined by *F(x)=18ℤ* and *G((0,y))={0}×6yℤ.* Then *(F,A)* is a soft ring over *R* and *(G,B)* is a soft ring over *R*$'$. Let *f:R→R*$'$ be defined by *f(x)={0,x}* and the function *g:A→B* be defined by *g(y)={0,3y).* Then *f* is a ring isomorphism, while *g* is a surjective map. For all *xϵA* we have *f(F(x))=f(18xℤ)=(0,18xℤ)* and *G(g(x))=G(0,3x)=(0,6(3x)ℤ)=(0,18xℤ).* Therefore *(f,g)* is a soft ring isomorphism and so *(F,A) ≃ (G,B).*

4. Conclusion

The concept of soft set theory was initiated by Molodtsov [11]. Maji et al [10] and Ali et al [3] introduced some operations on soft sets and gave some of their basic properties. Group and semi ring structures of soft sets were introduced by Aktas and Cagman [2] and Acar et al [1] respectively. This paper studied the ring structure and sub structures of the soft set theory and their algebraic properties. To extend this work, one could study the soft set theory applied to other algebraic structures such as fields and algebras.

References

[1]. U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Computers and Mathematics with Applications, 59(2010) 3458-3463.

[2]. H. Atkas and N. Cagman, Soft sets and soft groups, Information sciences, 1(2007) 2726-2735.

[3]. M. I Ali, F. Feng, X. Liu, W.K Min and M. Shabir, On some new operations in soft set theory, Computers and mathematics with Applications, 57(2009) 1547-1553.

[4]. A. O Atagun and A. Segzin, Soft substructures of rings, fields and modules, Computers and mathematics with Applications, 61(2011) 592-601.

[5]. N. Cagman and S. Enginoglu, Soft set theory and uni-int decision making, European Journal of Operational Research, 207(2010) 848-855.

[6]. Y. Celik, C. Ekiz and S. Yamak, A new view on soft rings, Hacettepe Journal of Mathematics and statistics, 40(2011) 273-286.

[7]. W. L Gau and D. J. Bueher, Vague sets, IEEE Trans. System Man Cybernet 23(1993) 610-614.

[8]. Y. B Jun, Soft BCK/BCI- algebras, Computers and Mathematics with Applications, 56(2008) 1408-1413.

[9]. P. K Maji, R.Biswas and A.R Roy, An application of soft sets in a decision making problem, Computers and mathematics with Applications, 44(2002) 107

[10]. P. K Maji, R. Biswas and A.R Roy, Soft set theory, Computers and Mathematics with Applications, 45(2003) 555-562.

[11]. D. Molodtsov, Soft set theory- First results, Computers and Mathematics with Applications, 37(1999) 19-31.

[12]. Z. Pawlak, Rough sets, International Journal of information and computer sciences, 11(1982) 341-356.

[13]. L. A. Zadeh, Fuzzy sets, information and control, 8(1965) 338-353.