Scattering Of Water Waves In A Deep Ocean In Presence Of An Inertial Surface In Front Of A Thin Floating Dock

Subhabrata Gangopadhyay, Uma Basu

Abstract: - The phenomenon of scattering of water waves is examined in presence of discontinuity at the free surface of a deep ocean. The surface discontinuity is thought of as originating due to the presence of a semi-infinite inertial surface on one half of the free surface and a thin floating semi-infinite dock on the other half. Appropriate expressions for Green's functions are set up for the fluid occupied in the inertial surface and for the fluid occupied below the thin dock. Employing Green's second integral theorem to the above mentioned Green's functions and the potential function the problem is reduced to finding out solutions of a pair of coupled Fredholm integral equation. An integral expression for the reflection coefficient is obtained.

Keywords: - Fredholm integral equation, Green's function, Green's second identity, inertial surface, reflection coefficient, semi-infinite dock, surface discontinuity.

1 INTRODUCTION

Problems concerning scattering of water waves arising out of a discontinuity in the free surface are relevant to ongoing research in the field of coastal dynamics and ocean engineering. The free surface discontinuity arises, for example, when half the surface of water is free and the remaining half is occupied by a dock extended upto infinity. Instead of free surface, the half can be covered by a semi-infinite intertial surface composed of a thin but uniform distribution of non-interacting floating materials, e.g. broken ice, floating mat, etc. It is clear that the discontinuity occurs because of two different types of boundary conditions on the two halves. The dock problem was mathematically formulated by Friedrich and Lewy[1]. Chung and Linton[2] found analytic expressions of hydrodynamic coefficients of scattering of waves across a finite gap between two semi-infinite elastic plates using techniques of residue calculus. The problem of water wave scattering in presence of finite or semi-infinite elastic plate have been dealt with in [3,4,5,6,7] using Weiner-Hopf techniques. Peters [8] applied Wiener-Hopf method to investigate water wave scattering on the surface of deep water considering the effect of a floating mat. Chakraborti[9] formulated a singular integral equation approach to derive expressions for reflection and transmission coefficients for the problem relating to a semi-infinite inertial surface. Chakraborti, Mandal and Gayen[10] employed Fourier analysis and singular integral equation to examine the semi-infinite dock problem.

- Subhabrata Gangopadhyay is currently an Associate Professor in the Department of Mathematics, The Bhawanipur Education Society College, 5, L.L.R. Sarani, Kolkata – 700 020, India. E-mail: subhagan31@gmail.com
- Uma Basu, is currently a Professor in the Department of Applied Mathematics, University of Calcutta, 92, A.P.C. Road. Kolkata – 700 009. India.

Hermans [11] applied integral equations to study free-surface waves interaction with a thick flexible dock. Mandal and De[12] investigated surface wave propagation over small undulation at the bottom of the ocean with surface discontinuity using an eigen function expansion method. The preset paper deals with the problem of scattering of water waves in presence of an semi-infinite inertial surface in front of a semi-infinite thin dock floating in deep water. The inertial surface is assumed to have uniform surface density. Appropriate Green's functions and velocity potential function in the regions occupied by water lead to integral expressions for the potential function in two different zones. Green's second identity is applied to the Green's functions and the potential function. The matching conditions along the vertical line of the surface of discontinuity suggest the continuity of pressure and velocity at the junction. Utilising the same, a pair of coupled Fredholm integral equation is arrived at, which can be solved by numerical methods. Since the water below the dock is undisturbed there will be no transmission of incoming wave train. An integral representation of the reflection coefficient is found out.

2 MATHEMATICAL FORMULATION

A two-dimensional potential flow in an ocean of infinite depth is considered. A rectangular cartesian co-ordinate system with y-axis vertically downwards along the depth of the ocean is chosen. The semi-infinite floating dock of small width and uniform density is assumed to occupy the region given by $0 \leq x < \infty, y = 0$ while the inertial surface is assumed to occupy the region given by $-\infty \leq x \leq 0, y = 0$. The line of discontinuity is along $x = 0$. The undisturbed upper surface is considered to be along $y = 0$. Let $\psi(x, y) = Re\{\varphi(x, y)e^{-i\omega t}\}$ represent the velocity potential of the fluid motion for the two-dimensional fluid region. The mathematical problem under consideration is to solve the boundary value problem in which function φ satisfies the following Laplace equation along with certain boundary conditions:

$$\nabla^2\varphi = 0 \text{ in the entire fluid region} \qquad 2.1$$

$$\varphi_y + K_1\varphi = 0 \text{ on } y = 0, x < 0 \qquad 2.2$$

$$\varphi_y = 0 \text{ on } y = 0, x > 0 \qquad 2.3$$

where $K_1 = \frac{K}{1-\epsilon_1 K}$ with $K = \frac{\omega^2}{g}$, ω is the angular frequency, g is the acceleration due to gravity.

The bottom boundary condition is given by:

$$\varphi, \nabla\varphi \to 0 \text{ as } y \to \infty \qquad 2.4$$

The far field behaviour of the potential function is described by:

$$\varphi(x,y) \sim \begin{cases} 0 & as\ x \to \infty \\ \varphi^1(x,y) + R\varphi^1(-x,y) & as\ x \to -\infty \end{cases} \qquad 2.5$$

where R is the unknown reflection coefficient, of the present scattering problem and

$$\varphi^1(x,y) = e^{iK_1 x - K_1 y} \qquad 2.6$$

3 METHOD OF SOLUTION

Let $G_1(x,y;\xi,\eta)$ be the Green's function due to a submerged line source at (ξ,η) for the region occupied by the inertial surface. Let the corresponding function for the water below the thin dock be $G_2(x,y;\xi,\eta)$. Then G_1 and G_2 satisfy the following conditions:

$$\nabla^2 G_{1,2} = 0 \text{ in the fluid region except at } (\xi,\eta);\ -\infty < x, \xi < \infty \qquad 3.1$$

$$\frac{\partial G_1}{\partial y} + K_1 G_1 = 0 \text{ on } y = 0, x < 0 \qquad 3.2$$

$$\frac{\partial G_2}{\partial y} = 0 \text{ on } y = 0, x > 0 \qquad 3.3$$

$$G_{1,2} \to lur \text{ where } r = \{(x-\xi)^2 + (y-\eta)^2\}^{\frac{1}{2}} \to 0 \qquad 3.4$$

$$\nabla^2 G_{1,2} \to 0 \text{ as } y \to \infty \qquad 3.5$$

and the radiation condition

$$G_1 \to -2\pi i e^{-K_1(y+\eta) + iK_1|x-\xi|} \text{ as } |x-\xi| \to \infty \qquad 3.6$$

The above boundary value problems for G_1 and G_2 have the following solutions as given by Thorne [13]:

$$G_1(x,y,;\xi,\eta) = ln\frac{r}{r\prime} - 2\int_C \frac{e^{-k(y+\eta)}}{k-K_1} \cos k(x-\xi)\, dk \qquad 3.7$$

where $r' = \{(x-\xi)^2 + (y+\eta)^2\}^{\frac{1}{2}}$. The path C is along the positive real axis in the complex k-plane having an indentation below the pole $k = K_1$.

$$G_2(x,y,;\xi,\eta) = ln\frac{r}{r\prime} - 2\int_{C\prime} \frac{e^{-k(y+\eta)}}{k} \cos k(x-\xi)\, dk \qquad 3.8$$

where the path C' is along the positive real axis in the complex k-plane having an indentation below the pole $k = 0$.

Equation (3.7) has the following alternative representation:

$$G_1(x,y;\xi,\eta) = -2\pi i e^{-K_1(y+\eta) + iK_1|x-\xi|} - 2\int_0^\infty \frac{(kcosky - K_1 sinky)(kcosk\eta - K_1 sink\eta)}{k(k^2+K_1^2)} \cdot e^{-k|x-\xi|} dk \qquad 3.9$$

Equation (3.8) has the following alternative representation

$$G_2(x,y;\xi,\eta) = -2\pi i - 2\int_0^\infty \frac{coskycosk\eta}{k} e^{-k|x-\xi|} dk \qquad 3.10$$

Next, we make use of Green's integral theorem to the function $\varphi - e^{iK_1 x - K_1 y}$ and $G_1(x, \leq 0, \xi \leq 0)$ in the region bounded by the lines $y = 0, -X \leq x \leq 0$; $x = -X,\ 0 \leq y \leq Y$; $y = Y, -X \leq x \leq 0$; $x = 0, Y \leq y \leq 0$; a small circle of radius ϵ and centre at (ξ,η). This leads to

$$2\pi\varphi(\xi,\eta) = 2\pi e^{-K_1\eta + iK_1\xi} + \int_0^\infty \left[\varphi(0,y)\frac{\partial G_1}{\partial x}(0,y;\xi,\eta) - G_1(0,y;\xi,\eta)\frac{\partial\varphi}{\partial x}(0,y)\right]dy + \chi(\xi,\eta);\ \xi \leq 0 \qquad 3.11$$

where
$$\chi(\xi,\eta) = \int_0^\infty \left[iK_1 e^{-K_1 y} G_1(0,y;\xi,\eta) - e^{-K_1 y}\frac{\partial G_1}{\partial x}(0,y;\ \xi,\eta)\right]dy \qquad 3.12$$

To evaluate $\chi(\xi,\eta)$, we once again make use of Green's integral theorem to the functions $\Omega(x,y) = e^{-K_1 y - iK_1 x}$ and $G_1(x,y;\xi,\eta)$ in the region mentioned above. We are led to

$$\chi(\xi,\eta) = 2\pi\Omega(\xi,\eta) - 2\int_0^\infty e^{-K_1 y}\frac{\partial G_1}{\partial x}(0,y;\ \xi,\eta)dy,\ \xi \leq 0 \qquad 3.13$$

Using equation (3.13) in the equation (3.12) and ultimately making $X, Y \to \infty, \epsilon \to 0$ we obtain

$$2\pi\varphi(\xi,\eta) = 2\pi e^{-K_1\eta}\left(e^{-iK_1\xi} + e^{iK_1\xi}\right) + \int_0^\infty \left[\varphi(0,y)\frac{\partial G_1}{\partial x}(0,y;\xi,\eta) - G_1(0,y;\xi,\eta)\frac{\partial\varphi}{\partial x}(0,y) - 2\pi e^{-K_1 y}\frac{\partial G_1}{\partial x}(0,y;\xi,\eta)\right]dy,\ \xi \leq 0 \qquad 3.14$$

Again employing Green's integral theorem to the functions φ and $G_2(x \geq 0, \xi \geq 0)$ in the region bounded by the lines $y = 0,\ 0 \leq x \leq X$; $x = X,\ 0 \leq y \leq Y$; $y = Y,\ 0 \leq x \leq X$; $x = 0, 0 \leq y \leq Y$; a small circle of radius ϵ at (ξ,η). Making $X, Y \to \infty, \epsilon \to 0$, the following integral expression for $\varphi(\xi,\eta)$ is arrived at

$$2\pi\varphi(\xi,\eta) = \int_0^\infty \left[G_2(0,y;\xi,\eta)\frac{\partial\varphi}{\partial x}(0,y) - \frac{\partial G_2}{\partial x}(0,y;\xi,\eta)\varphi(0,y)\right]dy,\ \xi \geq 0 \qquad 3.15$$

Now we consider the limiting values of the expressions (3.14) and (3.15) by letting $\xi \to 0-$ and $\xi \to 0+$ respectively. We get $\lim_{\xi\to 0+}\varphi(\xi,\eta)$ and $\lim_{\xi\to 0-}\varphi(\xi,\eta)$. Now, since φ is continuous along the line $x = 0$, taking average of the two expressions we have

$$\varphi(0,\eta) = e^{-K_1\eta} + \frac{1}{2\pi}\int_0^\infty (G_2 - G_1)(0,y;0,\eta)\frac{\partial\varphi}{\partial x}(0,y)dy \qquad 3.16$$

Again differentiating the equations (3.14) and (3.15) with respect to ξ we make $\xi \to 0\mp$ in the resulting expressions. Using the continuity of $\frac{\partial\varphi}{\partial x}$ along the line $x = 0$ and taking the average of $\lim_{\xi\to 0-}\frac{\partial\varphi}{\partial\xi}(\xi,\eta)$ and $\lim_{\xi\to 0+}\frac{\partial\varphi}{\partial\xi}(\xi,\eta)$ we have

$$\frac{\partial\varphi}{\partial\xi}(0,\eta) = ie^{-K_1\eta} + \frac{1}{2\pi}\int_0^\infty \frac{\partial^2}{\partial\xi\partial x}(G_1 - G_2)(0,y;0,\eta)\varphi(0,y)dy \qquad 3.17$$

Equations (3.16) and (3.17) are the two coupled Fredholm integral equations with regular kernels for the unknown functions $\varphi(0,\eta)$ and $\frac{\partial\varphi}{\partial\xi}(0,\eta)$. This pair of integral equations may be solved by standard numerical methods.

The hydrodynamic coefficient of interest in the current dock problem is the reflection coefficient R which is obtained by making $\xi \to -\infty$ in (3.14) and using expressions for G_1 and G_2 as given by (3.9) and (3.10).

$$R = \int_0^\infty e^{-K_1 y}\left[K_1\varphi(0,y) + i\frac{\partial\varphi}{\partial x}(0,y)\right]dy \qquad 3.18$$

After solving equations (3.16) and (3.17) for $\varphi(0,\eta)$ and $\frac{\partial\varphi}{\partial\xi}(0,\eta)$, these functions can be put in (3.18) to get a numerical estimate of $|R|$.

4 CONCLUSION

The problem of scattering of water waves in deep water in presence of an inertial surface in front of thin floating dock is explored by a novel method. After formulating the Green's functions for the two halves, one for the half occupied by the inertial surface and the other for the half occupied by the fluid below the dock, Green's second identity is made use of to the Green's functions and the potential function thereby resulting in two coupled Fredholm integral equations which may be computed by suitable numerical methods. The method is a simple one taking care of the continuity conditions of the potential function and its derivative along the vertical line of the surface of discontinuity. An integral representation of the coefficient of reflection of the incoming wave by the dock is arrived at. The present study is of practical relevance in its applications to coastal dynamics and dock construction engineering.

REFERENCES

[1] K.O. Friedrich and H.Lewy, The dock problem, Comm.Applied Maths 1, pp. 135-148 (1948)

[2] H.Chung, C.M.Linton, Reflection and transmission of waves across a gap between two semi-infinite elastic plates on water, J.Mech.Appl.Math 1 (1), pp. 1-15 (2004)

[3] D.V.Evans, T.V.Davies, Wate-ice interaction, Technical Report 1313, Stevens Institute of Technology (1968)

[4] R.V.Goldstein, A.V.Marchenko, The diffraction of plane gravitational waves by edge of an ice-cover, J.Appl. Math. Mech. (PMM), 53(6), pp. 731-736 (1989)

[5] N.J.Balmforth, R.V.Craster, Ocean waves and ice sheets, J.Fluid Mech., 395, pp. 89-124 (1999)

[6] H.Chung, C.Fox, Calculation of wave-ice interaction using the Wiener-Hopf technique, NZ J.Math., 31, pp. 1-18 (2002)

[7] T.D.William, M.H.Meylan, The Wiener-Hopf and residue calculus solutions for a submerged semi-infinite elastic plate, J.Eng. Math., 75, pp. 81-106 (2012)

[8] A.S.Peters, The effect of a floating mat on water waves, Commun Pure Appl. Math., 3, pp. 319-354 (1950)

[9] A.Chakraborti, On the solution of the problem of scattering of surface water waves by a sharp discontinuity in the surface boundary conditions, ANZIAM J. 42, pp. 277-286 (2000)

[10] A.Chakraborti, B.N.Mandal, R.Gayen, The dock problem revisited, Int.J.Math. Soc. 21, pp. 3459-3470 (2005)

[11] A.J.Hermans, Free-surface wave interaction with a thick flexible dock or very large floating platform, J.Eng.Math., 58, pp. 77-90 (2007)

[12] B.N.Mandal, Soumen De, Surface wave propagation over small undulation at the bottom of an ocean with surface discontinuity, Geophysical and Astrophysical Fluid Dynamics, 103, pp. 19-30 (2009)

[13] R.C.Thorne, Multiple expressions in the theory of surface waves, Math. Proc. Camb. Phil. Soc., 49, pp. 707-715 (1953)