Deterministic Mathematical Model Of Cholera, Predicting Chances Of Its Outbreak

Sani A. Fakai, M. O. Ibrahim, A. Danbaba

ABSTRACT: - Outbreaks of cholera occur suddenly, if not controlled, can spread like wild bush fire. In this work, a determistic mathematical model of cholera was developed from some modifications of previous cholera models. A system of three differential equations was used. Analysis was performed on the Jacobian matrix assuming zero Vibrio Cholerae environments. The basic reproduction number R_o was obtained as $\frac{\varepsilon\alpha p}{k(\gamma+\tau)(g-l+\omega)}$ and the critical number or threshold S_c was also obtained as $\frac{k(\gamma+\tau)(g-l+\omega)}{\varepsilon\alpha}$. These two values are used to predict occurrence of cholera outbreak in a community. Zero equilibrium state is stable when $R_o < 1$ and unstable when $R_o > 1$, these conditions are explained in respect to the model's parameters.

Key words: - Vibrio Cholerae, therapeutic, threshold, toxigenic, hygienic, basic reproduction number, contaminated.

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INTRODUCTION

Cholera is said to be the epidemic that urge health education in the early nineteenth century, due to its contagious nature which if allow deteriorating, it has no respect to class of people, and it is still maintaining this threatening gesture. Mathematical modelling being the mainstay of epidemiological theory, is crucial to apply in studying cholera dynamics because ability to model disease dynamics can be used to forecast the danger of a major epidemic. Hartley et al (2006) stated that "Cholera is an ancient disease that continues to cause epidemic and pandemic disease despite on-going effort to limit its spread. Mathematical models provide one means of assessing the utility of various proposed interventions". Since a good understanding of the transmission, dynamics and ecology of cholera in emergent regions can help to improve the control of future epidemics, mathematical models provide a quantitative and potentially valuable tool for this purpose. In its extreme manifestation, cholera is one of the most rapidly fatal illnesses known. A healthy person may become hypertensive within an hour of the onset of symptoms and may die within 2 – 3 hours if no treatment is provided. Categorically if infected, the disease progress from the first liquid stool to shock in 4 – 12 hours with death following in 18hours to several days. Cholera cases were categorised by Akor (2007) as "10% of cholera cases present with copious painless, watery diarrhoea (rice-water diarrhoea or stool) and are associated with high mortality rate. The remaining 90% have either mild diarrhoea no symptoms at all. Vomiting may occur in 80% of cases and abdominal pain and muscle cramps, shock, and renal failure are features of severest infection".

Typical cholera begins over quite a short period of time with stomach pains without feeling sick. There may also be a mild fever, diarrhoea and vomiting begins and may continue for several hours. This is followed by copious, watery diarrhoea that is pale and flaky and looks like rice-water. The fluid loss may be as high as one litre per hour. Rapid loss of enormous quantity of fluid and electrolytes lead to intensive dehydration with muscular cramps. The skin becomes cold dummy and rankled and the eyes become sunken. The urine output reduces drastically and the patient becomes mentally alert. Blood pressure falls and the pulse sound can be heard. If the fluids and electrolytes are not replaced immediately the patient can die instantly due to acute circulatory failure. About one hundred million *Vibrio Cholerae* bacteria must be typically ingested to cause cholera disease in a normal healthy adult. It may not reach that amount in people with lower gastric acidity. According to Isere and Osemwenkhae (2010) "Experiment suggests that a dose of 10^6 cells/ml is expected to cause cholera outbreak". Yoon and Mekalano (2006) "It is tempting to speculate in ORS may have contributed to an vivo selection of El Tor strains, at least in those patients that were not simultaneously treated with antibiotics". "Because the infectious dose of *V. Cholerae* is known to be quite high for humans with normally acidic contents, transmission via contaminated food may be more important than transmission through contaminated water alone. The objective of this research study is to marry the ideas and pattern adopted by Codeco (2001), Wang and Liao (2010), Mukandavire et al (2010) and Hartley et al (2006). To produce a harmonized model with four differential equations that change to three differential equations by dropping recovered population's equation $\frac{dR}{dt}$, which is assumed to have merged with the susceptible equation $\frac{dS}{dt}$. Two vital parameters were incorporated to predict the dynamics of cholera outbreak are: 1.The rate of therapeutic treatment of cholera disease denoted by τ. and 2. The rate of reduction of *Vibrios Cholerae* bacteria as a result of sanitation and hygienic measures taken by people is denoted by ω. If no immunity, recovered people easily become susceptible after infection. Recovery equation has less effect in the model; hence recovery equation is dropped as in Tian, et al (2010). Analysis will be conducted on equations 4, 5 and 6.

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THE MODEL'S EQUATIONS

The dynamics of the disease is given by a system of differential equations below:

$$\frac{dS}{dt} = np - \alpha h(B)S - nS \qquad 1$$

$$\frac{dI}{dt} = \alpha h(B)S - (\gamma + \tau)I \qquad 2$$

$$\frac{dB}{dt} = \varepsilon I - (g - l + \omega)B \qquad 3$$

S is the susceptible and it comprises all the individuals in the community. The individuals are either born in the community or they immigrated into it at the rate *np,* susceptible S become infected with cholera at the rate α, infected I contribute to the population of *Vibrio Cholerae* at a rate ε, infected people either die from cholera disease or recover at a rate γ. Recovered people become susceptible again, since it is assumed that no immunity after infection. Chance of catching cholera as a result of contact with contaminated food or water is *h(B)*. Equation 1 describe the dynamics of susceptible in the community with size *p.* Susceptible individuals are recruited at the rate *np,* they became infected at the rate α*h(B),* where α is the rate of consuming contaminated food or water. $h(B) = \frac{B}{k+B}$ where B is the concentration of toxigenic *Vibrio Cholerae* bacteria in food and water and *k* is the concentration of *Vibrio Cholerae* in food and water that yields 50% chance of catching cholera disease. Equation 2 describes the dynamics of infected people in the community. Infected people increase as susceptible become infected and their number decreases as the infected recover or die as a result of cholera disease. Also measures to curtail the spread of the disease, such as hygiene and total sanitation indirectly affect infections. Equation 3 describes the dynamics of pathogenic *Vibrio Cholerae* in the environment, comprising the contaminated food or water consumed by people. The total population at time t, denoted by *p*(t) is given by

p(t) = S(t) + I(t) + R(t).

Substituting $\frac{B}{k+B}$ for *h*(B), We have:

$$\frac{dS}{dt} = np - \frac{\alpha BS}{k+B} - nS \qquad 4$$

$$\frac{dI}{dt} = \frac{\alpha BS}{k+B} - (\gamma + \tau)I \qquad 5$$

$$\frac{dB}{dt} = \varepsilon I - (g - l + \omega)B \qquad 6$$

As in Isere and Osemwenkhae (2010), with S = *p*, I = B = 0, equation 4 becomes linear differential equation:

$$\frac{dS}{dt} = np - \frac{\alpha BS}{k+B} - nS$$

$$\Rightarrow \quad \frac{dS}{dt} = np - nS$$

$$\frac{dS}{dt} + nS = np$$

$$\int e^{nt} = \int npe^{nt}dt + c$$

$$= \int pe^{nt} + c$$

$$\textbf{If } c = (S - p)e^{-nt}$$

$$S(t) = p + (S - p)e^{-nt}$$

Where S_i is the initial number of susceptible individuals
As t→∞, $S_i = p$ then S →p

this is the asymptotic population size. It proves the attainment or achievement of disease free equilibrium. As in Bayleyegn (2009), in analysing the model, the Jacobian matrix will now be:

$$J_E = \begin{bmatrix} -n - \frac{\alpha B}{k+B} & 0 & \frac{\alpha S}{k+B} - \frac{\alpha BS}{(k+B)^2} \\ \frac{\alpha B}{k+B} & -(\gamma - \tau) & -\frac{\alpha S}{k+B} + \frac{\alpha BS}{(k+B)^2} \\ 0 & \varepsilon & -(g - l + \omega) \end{bmatrix} \dots\dots 7$$

J_E is the Jacobian of equilibrium variables.

STABILITY ANALYSIS

Now if a small number of infective is introduced into the community, will there be disease Free State? From Akor (2007) only 10% cholera cases comes with diarrhoea and vomiting, but all infected people can contribute to the increase and spread of *Vibrio Cholerae* in the environment. When there are no *Vibrio Cholerae* bacteria in community, B = 0, nobody is infected with cholera, I = 0 and the population remains S = *p*. Analysing system of the model using these values, we start by obtaining the determinant of J_E at S = *p* and I = B = 0, and find the roots of the characteristic equation thus:

$$\text{Det.J}(p,0,0) = \begin{vmatrix} -n - \lambda & 0 & \frac{-\alpha p}{k} \\ 0 & -((\gamma + \tau) - \lambda) & \frac{\alpha p}{k} \\ 0 & \varepsilon & -((g - l + \omega) - \lambda) \end{vmatrix} = 0$$

(- n - λ)[(-(γ+τ)-λ)(-λ-(g-l+ω)) - (εα*p*)/k] – 0 + 0 = 0…………………8

Clearly λ_1 = -n

and   (-γ - τ - λ)(-λ - g + l - ω) - $\frac{\varepsilon\alpha p}{k}$ = 0 …………………..9

$k\lambda^2 + k\lambda(\gamma+\tau+g-l+\omega) + k(\gamma(g-l+\omega) + \tau(g-l+\omega)) - \varepsilon\alpha p = 0$

$k\lambda^2 + k\lambda(\gamma+\tau+g-l+\omega) + k(\gamma+\tau)(g-l+\omega) - \varepsilon\alpha p = 0$

$$\lambda = \frac{-k\,(\gamma+\tau+g-l+\omega) \pm \sqrt{[k(\gamma+\tau+g-l+\omega)]^2 - 4k(k(\gamma+\tau)(g-l+\omega) - \varepsilon\alpha p)}}{2k}$$

$$= \frac{-k\,(\gamma+\tau+g-l+\omega) \pm \sqrt{4k\varepsilon\alpha p - 4k(k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2}}{2k}$$

$$\lambda = \frac{-k\,(\gamma+\tau+g-l+\omega) \pm \sqrt{4k(\varepsilon\alpha p - (k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2}}{2k}$$

$$\lambda_2 = \frac{-k\,(\gamma+\tau+g-l+\omega) + \sqrt{4k(\varepsilon\alpha p - (k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2}}{2k}$$

$$\lambda_3 = \frac{-k\,(\gamma+\tau+g-l+\omega) - \sqrt{4k(\varepsilon\alpha p - (k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2}}{2k}$$

For λ_2 to be negative so that the disease-free equilibrium to be asymptotically stable, such that the disease dies out with time,

$$\sqrt{4k(\varepsilon\alpha p - k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2} \; < k(\gamma+\tau+g-l+\omega) ....10$$

THE BASIC REPRODUCTION NUMBER R_o AND THE CRITICAL NUMBER (TRESHOLD) S_c OF THE MODEL

We have established that:

$$\sqrt{4k(\varepsilon\alpha p - k(\gamma+\tau)(g-l+\omega)) + [k(\gamma+\tau+g-l+\omega)]^2} < k(\gamma+\tau+g-l+\omega)$$

For the disease to dies out with time. Hence if

$$\sqrt{4k\varepsilon\alpha p - 4k^2(\gamma+\tau)(g-l+\omega) + [k(\gamma+\tau+g-l+\omega)]^2} \; < k(\gamma+\tau+g-l+\omega)$$

we observe that

$4k\varepsilon\alpha p - 4k^2(\gamma+\tau)(g-l+\omega) < 0$ for the condition in equation (10) to hold.

$$\left[\frac{4k\varepsilon\alpha p}{4k^2(\gamma+\tau)(g-l+\omega)} - \frac{4k^2(\gamma+\tau)(g-l+\omega)}{4k^2(\gamma+\tau)(g-l+\omega)}\right]4k^2(\gamma+\tau)(g-l+\omega) < 0$$

$$= \left(\frac{4k\varepsilon\alpha p}{4k^2(\gamma+\tau)(g-l+\omega)} - 1\right)4k^2(\gamma+\tau)(g-l+\omega) < 0$$

Since $4k^2(\gamma+\tau)(g-l+\omega) > 0$

Therefore $\left(\frac{4k\varepsilon\alpha p}{4k^2(\gamma+\tau)(g-l+\omega)} - 1\right) < 0$

This implies $\frac{\varepsilon\alpha p}{k(\gamma+\tau)(g-l+\omega)} - 1 < 0$

Suppose $\frac{\varepsilon\alpha p}{k(\gamma+\tau)(g-l+\omega)} = R_O$..............11

We get $R_o - 1 < 0$

This means $R_o < 1$

Therefore the basic reproduction number of the model is

$R_o = \frac{\varepsilon\alpha p}{k(\gamma+\tau)(g-l+\omega)}$

The basic reproduction number R_o was defined by Fraser et al (2011) as: "The number of secondary infections generated by a primary infection in susceptible population and which thus measure the intrinsic transmissibility of an infectious agent. For an epidemic to expand in the early stage of spread more than one secondary case has to be generated by the primary case and hence we need

$R_o > 1$".

Theorem 1: The disease-free equilibrium state of this model is locally asymptotically stable if $R_o < 1$ and unstable if $R_o > 1$. From theorem 1, the disease-free equilibrium state is reached, since R_o obtained in this model is less than one, it means there is a unique disease-free equilibrium state and zero equilibrium state is locally asymptotically stable. It

means that when S = p, I = B = 0, there will be no cholera outbreak. The critical number or threshold, denoted by S_c of this model is $\frac{p}{R_o}$ which means $S_c = \frac{k(\gamma+\tau)(g-l+\omega)}{\varepsilon\alpha}$ These values R_o and S_c are used to predict and estimate cholera outbreak as follows: If the number of susceptible in the population is greater than the critical number S_c, an outbreak of cholera will occur, the outbreak can be endemic or pandemic. If the number of susceptible in the population is less than the critical number S_c, the number of cases of cholera will decrease and will return to zero. When $R_o < 1$, the stability of the disease-free equilibrium state is stable. When $R_o > 1$ the stability of disease-free equilibrium is unstable, that's stability of cholera epidemic is stable.

CONCLUSION

In our findings, we were made to know that vibrios cholerae bacteria, is not only ingested from drinking water, but also in food poorly handled interms hygienic and sanity conditions. Rate of growth of vibrios cholerae in the environment will increase, if sanity and hygienic measures decreases. In the analysis, the stability of zero equilibrium state is obtained. Disease-free equilibrium is achieved at zero infectives and zero toxigenic bacteria nunber. Critical number (threshold) S_c is obtained, which is one of the yard stick of predicting cholera outbreak. If the number of susceptible in the population is greater than the critical number (S_c) cholera outbreak will occur. Otherwise, the cases of cholera will decrease and return to zero. Another yard stick for predicting cholera outbreak in this work, is the basic reproduction number (R_o). If R_o is greater than one, cholera outbreak will occur in the concern community, if it is less than one the case(s) will reduce(s) and die(s) out. If R_o = 1, the case is undetermined. From this work if ω and τ are high this will result in decrease in g. The R_o will be less than one and the S_c will be greater than the susceptible. Which implies no cholera outbreak. If ω and τ are low g will be high $R_o > 1$ and S_c will be less than the susceptible which means cholera outbreak will occur. There is need to assertain the minimum rates of ω and τ which must be maintained to assure hinderance of cholera outbreak. It should be noted that cholera outbreak can occur in a region where it had never occured before, the chance of this happening is higher when $R_o > 1$.

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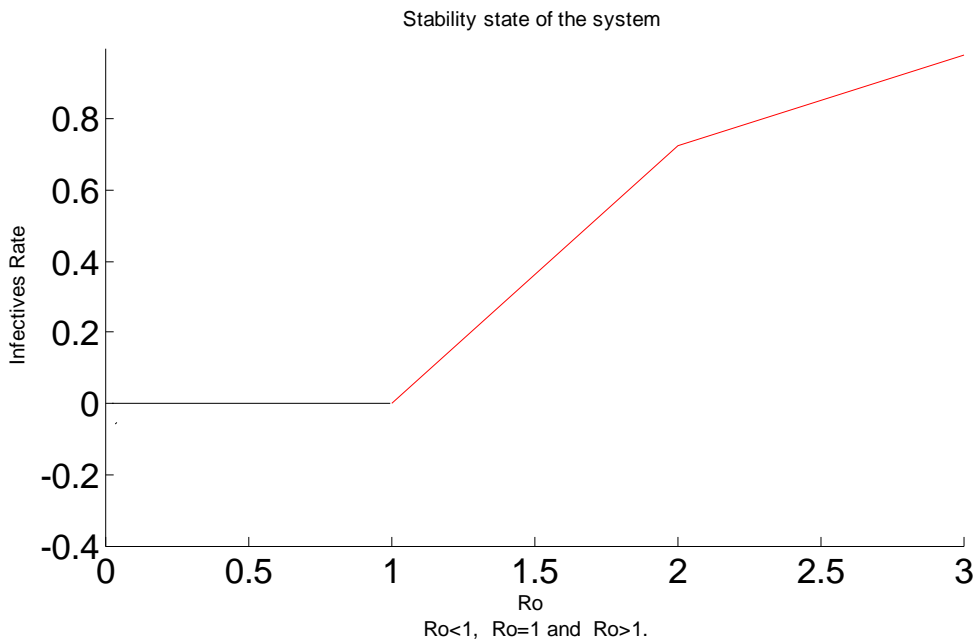
Figure 1.. Describing $R_o < 1$, the infection is zero,
$R_o = 1$, turning point,
$R_o > 1$, there will be outbreak
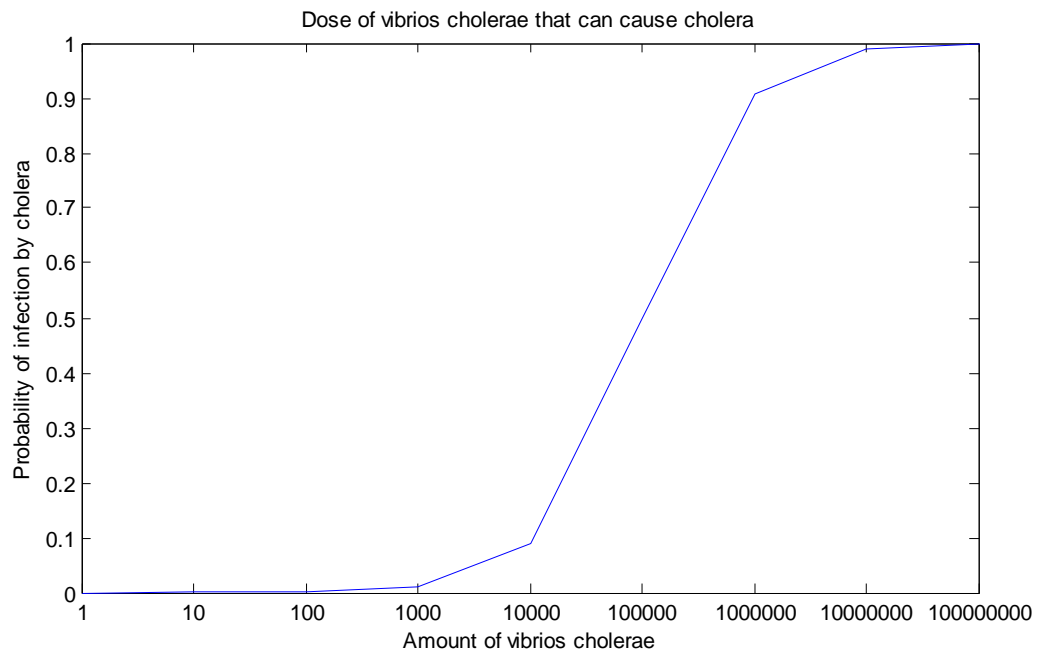
Figure 2.. Describing amount of vibrios cholerae that when ingested can cause the corresponding probability of catching cholera disease.