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Solutions of the Acoustic Problem in the 3D Form of the Helmholtz Equation Using DRBEM

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Abstract

This paper presents to the solutions of the acoustic problem using dual reciprocity boundary element method (DRBEM). The acoustic propagates inside the silencer and its sound can be formulated by the three dimensional (3D) Helmholtz equation. The form of the equation is which can be solved by the integral form on the boundary. The conventional boundary element method (BEM) is not suitable for solving the acoustic problems of silencers with higher Mach number subsonic flow, due to the presence of domain integral. The dual reciprocity method (DRM) is a method that converts the domain integral into the boundary integral. Mathematical formulations, discretization form and evaluation of the integrals are described and discussed. Algorithm of the method is also presented.

1. Introduction

Acoustic problem will influence the sound propagation inside the silencer system. In mathematical modeling of such problem, a three-dimensional Helmholtz equation can be employed to predict such a problem. The DRBEM method is an effective and powerful method, and has been developed to predict those problems [1].

Since the integral form of the equation has domain integral that is weak problem in the BEM. To change the domain integral into the boundary integral, Nardini and Brebbia was first introduced the DRM method [2]. After that, some researchers employed this method to solve the Poison and Helmholtz equations. Ramachandran described this method on his textbook with full program and some examples [3]. Zakerdoost and Ghassemi presented this method to the steady state convection-diffusion-radiation problems [4]. Lee, et al [5] used the DRBEM to model the acoustic radiation in a subsonic non-uniform flow field, and indicated that the Sommerfeld-radiation condition at infinite was satisfied when DRBEM was used to deal with this problem.

In the real fluids, the sound wave will attenuate gradually as propagation due to the acoustic damping effect, and part of the acoustic energy is transformed into the heat energy. Dokumaci developed a boundary integral method, and applied it to solve radiation problems in viscous dissipation medium [6]. Kara and Ben Tahar applied the boundary integral method to solve radiation problems in viscothermal dissipation medium [7]. The objectives of the present study are to derive the acoustic governing equation in three-dimensional potential flow with consideration of acoustic damping effect, and then to apply the boundary element method to predict the acoustic attenuation performance of acoustic problem in the silencer system with considerations of the 3D potential flow and acoustic damping effects.

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The following sections are organized as follows. Section 2 is described the physical problem and obtain the governing equation. Section 3 is given the DRM method that how to convert the domain integral into the boundary integral. The evaluation integrals are described in section 4 using numerical Gaussian quadrature method for regular integrals and analytical method for the singular integrals. Section 5 presents the algorithm of the DRBEM and finally conclusions are given in Section 6.

2. Physical Problem And Governing Equation

The complete dynamic basic equations in the fluid are the mass continuity, Navier-Stokes and energy equations [8]:

Mass continuity

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla . u = 0. \tag{1}$$

Navier-Stokes

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p + (\frac{4}{3}\mu + \mu_v)\nabla(\nabla . u). \tag{2}$$

Energy

$$\frac{dS}{dt} = 0. \tag{3}$$

State

$$p = c_0^2 \rho \tag{4}$$

where and are the ambient and excess density, respectively. u is the particle velocity, and p and are ambient sound velocity and sound pressure resulted from the acoustic disturbance, respectively; S is the entropy per unit mass; μ and μv are the coefficients of shear viscosity and volume viscosity, respectively.

Substituting Eq. (4) into Eq. (1) yields

$$\frac{1}{c_0^2}\frac{\partial p}{\partial t} + \rho_0 \nabla . u = 0. \tag{5}$$

Substituting Eq. (5) into Eq. (1) yields

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p + (\frac{4}{3}\mu + \mu_v)\nabla(\frac{1}{\rho_0 c_0^2}\frac{\partial p}{\partial t}). \qquad (6)$$

Differentiating Eq. (5) with respect to time *t* and Eq. (6) with respect to coordinates, and substituting yields

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial^2 t} + (\frac{4}{3}\mu + \mu_v)\nabla^2(\frac{1}{\rho_0 c_0^2}\frac{\partial p}{\partial t}) = 0. \qquad (7)$$

Using total derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + V_0.\nabla,$, Eq. (7) can be written as

$$\left[1 + \frac{1}{\rho_0 c_0^2}(\frac{4}{3}\mu + \mu_v)\frac{\partial}{\partial t}\right]\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial^2 t}$$
$$-2\frac{\partial}{\partial t}(V_0.\nabla)p - \frac{1}{c_0^2}(V_0.\nabla)(V_0.\nabla)p \qquad (8)$$
$$+\frac{1}{\rho_0 c_0^2}(\frac{4}{3}\mu + \mu_v)\nabla^2(V_0.\nabla)p = 0.$$

The particle velocity *u* and the sound pressure *p*
can be expressed in terms of the acoustic velocity potential ϕ as

$$u = -\nabla\Phi \qquad 9)$$

And pressure is expressed as

$$p = \rho_0\left[\frac{\partial\phi}{\partial t} + V_0.\nabla\phi - (\frac{4}{3}\mu + \mu_v)\nabla^2\phi\right]. \qquad (10)$$

Then, the Eq. (8) can be defined in terms of total velocity potential (Φ) as

$$\left[1 + \frac{1}{\rho_0 c_0^2}(\frac{4}{3}\mu + \mu_v)\frac{\partial}{\partial t}\right]\nabla^2\Phi - \frac{1}{c_0^2}\frac{\partial^2\Phi}{\partial^2 t}$$
$$-2\frac{\partial}{\partial t}(M_0.\nabla\Phi) - \frac{1}{c_0^2}(M_0.\nabla(M_0.\nabla\Phi) \qquad (9)$$
$$+\frac{1}{\rho_0 c_0^2}(\frac{4}{3}\mu + \mu_v)\nabla^2(M_0.\nabla\Phi) = 0.$$

where $M_0 = V_0 / c_0$ is Mach number of the 3D potential flow.

For the steady state harmonic motion, the acoustic velocity motion can be expressed as

$$\Phi = \phi e^{j\omega t} = \phi(\cos\omega t + j\sin\omega t) \qquad (10)$$

where $\omega = 2\pi f$ is the angular frequency and f is the frequency and $j = \sqrt{-1}$.

Finally, the governing equation (9) can be written as

$$\nabla^2\phi + k^2\phi = b(\phi), \ in \ \Omega$$
$$B.C. = Known, \quad on \ \Gamma \qquad (11)$$

where:

$$\begin{cases} b(\phi) = \frac{k^2}{k_0^2}\left[2j(M_0.\nabla\phi) + (M_0.\nabla)(M_0.\nabla\phi)\right] \\ k = \frac{k_0}{\sqrt{1 + \frac{jk_0}{\rho_0 c_0}(\frac{4}{3}\mu + \mu_v)}} \\ k_0 = \omega / c_0 \end{cases} \qquad (12)$$

k_0 is the wave number.

Eq. (11) is the governing equation for the sound field in the three-dimensional flow.

3. Dual Reciprocity Method (Drm)

In the boundary element method, for a body of the boundary Γ and domain Ω (as shown in Figure 1), the integral formulation of the Eq. (11) may be expressed as

$$e(p)\phi(p) = \int_\Gamma (G\frac{\partial\phi}{\partial n} - \phi\frac{\partial G}{\partial n})ds - \int_\Omega Gb(\phi)d\Omega \qquad (13)$$

where:

$$e(p) = \begin{cases} 1 & for\ P\ outside\ \ Surface \\ 0.5 & for\ P\ on\ \ Surface \\ 0 & for\ P\ outside\ \ Surface \end{cases} \qquad (14)$$

And G is the *Green's function* of the Helmholtz equation. For the 3D problem,

$$G = \frac{\exp(-jkr)}{4\pi r} \qquad (15)$$

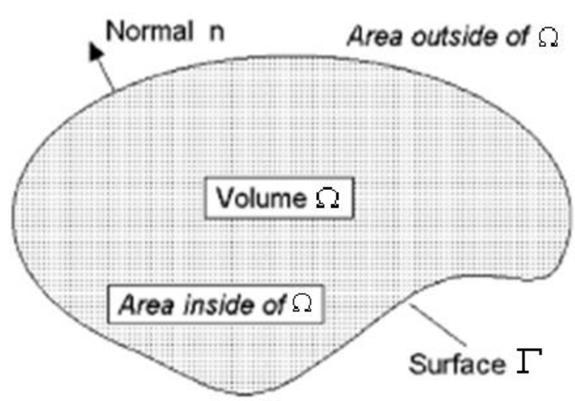

Fig: 1 Definition sketch of boundary domain

Eq. (13) contains the volume integral, which is difficult problem. Therefore, in order to overcome to this difficulty, one way may be converted to the boundary integral is the DRM [2-3]. This method is focused to the term $b(\phi)$ which may be approximated by the following expression:

$$b(X) = \sum_{i=1}^{N+L} f_i\,\alpha_i \qquad (16)$$

where α_i, f_i are interpolation coefficients and *radial basis function (RBF)*, respectively. N is the number of collocation nodes on the boundary, L is the number of collocation points in the domain and r_i is defined as the distance between the node under consideration and the node *i*.

For each simple source function f_i a particular solution h_i needs to be found and satisfied as

$$\nabla^2 h_i + k^2 h_i = f_i \tag{17}$$

One of the key ingredients of the DRM is the expression introduced in Eq. (16). There are many functions of the RBF to choose. It is usually chosen as

$$f_i = 1 + r_i \tag{18}$$

The particular solution of the Eq. (17) can be found as follows

$$(\nabla^2 + k^2) h_i = 1 + r_i \tag{19}$$

Since this solution is axisymmetric with respect to the source, it is independent of the polar angle θ, and thus Eq. (19) becomes [9]

$$\left(\frac{d^2}{dr_i^2} + \frac{2}{r_i}\frac{d}{dr_i} + k^2\right) h_i = 1 + r_i \tag{20}$$

A regular solution of the Eq. (20) is obtained [9-10]

$$h_i = \frac{1 + r_i}{k^2} - \frac{2}{k^4}\left(\frac{1 - \cos(kr_i)}{r_i}\right) \tag{21}$$

where:

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \tag{22}$$

Substituting of Eqs. (16) and (17) into Eq. (13) yields

$$\begin{aligned} e(p)\phi(p) &= \int_\Gamma \left(G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right) ds \\ &- \sum_{i=1}^{L+N} \alpha_i \int_\Omega G\left(\nabla^2 h_i + k^2 h_i\right) d\Omega \end{aligned} \tag{23}$$

Using the Green's identity, for two functions of the G and h_i, integral of the last term of the Eq. (23) can be expressed in terms of the boundary integral.

$$\begin{aligned} &\int_\Omega G\left(\nabla^2 h_i + k^2 h_i\right) d\Omega = \\ &e(p) h_i(p) - \int_\Gamma \left(G\frac{\partial h_i}{\partial n} - h_i\frac{\partial G}{\partial n}\right) ds \end{aligned} \tag{24}$$

The term $\frac{\partial h_i}{\partial n}$ is the normal derivative of the h_i and can be expressed as

$$h'_n = \frac{\partial h_i}{\partial n} = \frac{\partial h_i}{\partial r}\frac{\partial r}{\partial n} = \frac{\partial h_i}{\partial r}\frac{\vec{r}.\vec{n}}{r} \tag{25}$$

Finally, Eq. (23) can be expressed as

$$\begin{aligned} e(p)\phi(p) &= \int_\Gamma \left(G\frac{\partial \phi}{\partial n} - \phi\frac{\partial G}{\partial n}\right) ds \\ &- \sum_{i=1}^{N+L} \alpha_i \left\{ e(p) h_i(p) - \int_\Gamma \left(G h'_i - h_i\frac{\partial G}{\partial n}\right) ds \right\} \end{aligned} \tag{26}$$

Discretization form of the Eq. (26) can be represented as follows:

$$\begin{aligned} e_l \phi_l &= \sum_{j=1}^{N} L_{lj}\phi_{n_j} - \sum_{j=1}^{N} \hat{H}_{lj}\phi_j \\ &- \sum_{i=1}^{N+L} \alpha_i \left\{ e_l h_{li} - \sum_{j=1}^{N} L_{lj} h'_{ji} - \sum_{j=1}^{N} \hat{H}_{lj} h_{ji} \right\} \end{aligned} \tag{27}$$

where L_{lj} and $\hat{H}_{lj}$ are influence coefficients and defined as follows:

$$\begin{aligned} L_{lj} &= \int_\Gamma G ds \\ \hat{H}_{lj} &= \int_\Gamma \frac{\partial G}{\partial n} ds \end{aligned} \tag{28}$$

These integral can be evaluated by numerical and analytical methods.

Moreover setting

$$H_{lj} = 0.5\delta_{lj} + \hat{H}_{lj} \tag{29}$$

where δ_{lj} is the Kronecker delta, which is defined as $\delta_{lj} = 0$ for $l \neq j$ and $\delta_{lj} = 1$ for $l = j$, Eq. (27) may further be written as

$$\sum_{j=1}^{N} H_{lj}\phi_j = \sum_{j=1}^{N} L_{lj}\phi_{n_j} - \sum_{i=1}^{N+L} \alpha_i \left\{ \sum_{j=1}^{N} H_{lj} h_{ji} - \sum_{j=1}^{N} L_{lj} h'_{ji} \right\} \tag{30}$$

The last term of the RHS of the Eq. (30) can be calculated by the integral on the boundary. If the Neumann boundary condition employs (it means that ϕ_{n_j} is known) all terms of the RHS will be completely known. Then, Eq. (30) can be arranged as

$$\sum_{j=1}^{N} [H]_{lj} \{\phi\}_j = \{B\}_j \tag{31}$$

Eq. (31) may be solved and the unknown variables are obtained [11].

4. Evaluation Of The Integrals

In order to evaluate the boundary integrals appearing in Eq. (28), a numerical integration strategy should be performed. To do this, the boundary S of the domain is discretized into a sufficient number of elements as it is the first stage of the boundary or finite element methods.

The regular integrals of the Eq. (28) are evaluated numerically using a standard Gaussian quadrature, but for the singular integrands, an analytical method is employed to determine the integrals.

Inserting the Green's function of Eq. (15) into the Eq. (28), we have

$$L_{lj} = \int_{\Gamma} \frac{\exp(-jkr)}{4\pi r} ds$$
$$\hat{H}_{lj} = \int_{\Gamma} \frac{\partial}{\partial n}\left(\frac{\exp(-jkr)}{4\pi r}\right) ds \quad (32)$$

where r distance between the source point and integral element i.e. expressed as

$$\vec{r} = \vec{q} - \vec{p} \quad (33)$$

Figure 2 shows the integration element and source point.

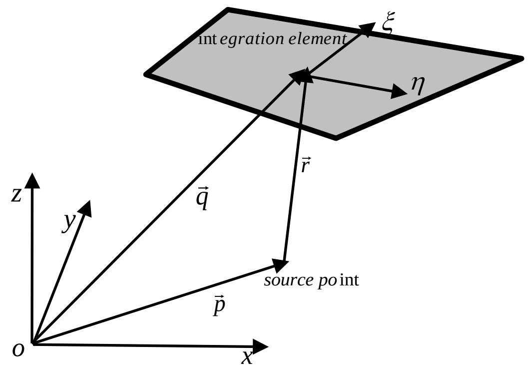

Fig: 2 Coordinate system of source point and integration element

For 3D geometry, linear or quadratic elements are adopted. In each element, the global coordinate $\vec{q}(\xi,\eta)$ is interpolated between the coordinates $\left(\vec{q}^{\beta}\right)$ of the element nodes through interpolation functions (where β is the node index).

$$q(\xi,\eta) = \sum_{\beta=1}^{4} N_{\beta}(\xi,\eta)\vec{q}^{\beta},\ linear$$
$$q(\xi,\eta) = \sum_{\beta=1}^{8} N_{\beta}(\xi,\eta)\vec{q}^{\beta},\ quadratic \quad (34)$$

ξ and η are intrinsic coordinates both taking a value between −1 to +1, and $N_{\beta}(\xi,\eta)$ are the shape functions which can be expressed as

$$\begin{cases} N_{\beta}(\xi,\eta) = \frac{1}{4}(1+\xi_{\beta}\xi)(1+\eta_{\beta}\eta), & \beta = 1\text{-}4 \quad linear \\ \begin{cases} N_{\beta}(\xi,\eta) = \frac{1}{4}(1+\xi_{\beta}\xi)(1+\eta_{\beta}\eta)(-1+\xi_{\beta}\xi+\eta_{\beta}\eta), & \beta = 1\text{-}4 \\ N_{\beta}(\xi,\eta) = \frac{1}{2}(1+\xi_{\beta}\xi+\eta_{\beta}\eta)\left(1-(\xi_{\beta}\eta)^2-(\eta_{\beta}\xi)^2\right), & \beta = 5\text{-}8 \end{cases} & quadratic \end{cases} \quad (35)$$

where η and ξ are nodal values of η_{β} and ξ_{β} (Figure 3). Two orthogonal tangential vectors are defined at point $\vec{q}(\xi,\eta)$ of the boundary as

$$\vec{a}_1 = \frac{\partial \vec{q}(\xi,\eta)}{\partial \xi} = \sum_{\beta=1}^{m} \frac{\partial N_{\beta}(\xi,\eta)\vec{q}}{\partial \xi}$$
$$\vec{a}_2 = \frac{\partial \vec{q}(\xi,\eta)}{\partial \eta} = \sum_{\beta=1}^{m} \frac{\partial N_{\beta}(\xi,\eta)\vec{q}}{\partial \eta} \quad (36)$$

where $m = 4\,(\text{linear})\,\text{or}\,8\,(\text{quadratic})$ locally numbers the nodes within each element.

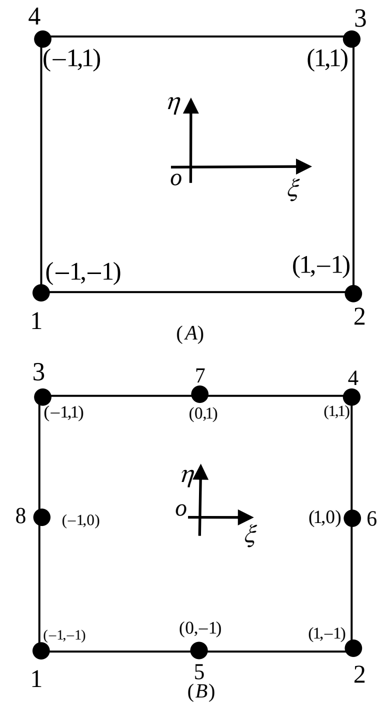

Fig: 3 Nodes over a quadrilateral element, (A) linear, (B) quadratic

After transformation from the global coordinates to the intrinsic coordinate, the Eq. (31) can be written as

$$L_{lj} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{\exp(-jkr)}{4\pi r} \left|J(\xi,\eta)\right| d\xi d\eta$$
$$\hat{H}_{lj} = \int_{-1}^{+1}\int_{-1}^{+1} \frac{\partial}{\partial r}\left(\frac{\exp(-jkr)}{4\pi r}\right)\frac{\partial r}{\partial n}\left|J(\xi,\eta)\right| d\xi d\eta \quad (37)$$

These above integrals can be determined by numerical Gaussian quadrature or analytical methods [12].

5. Algorithm of the DRBEM

Boundary element method (BEM) is powerful numerical tool for solving many engineering problems. First step should be defined the body geometry into some collocation points or elements. In the BEM, the solutions of the Poison or Helmholtz equations are expressed in the three integral forms in which one of them are in domain integrals. For this reason, DRM is employed to convert this domain integral into the boundary integral using RBF and internal nodal. Figure 4 shows the flowchart algorithm of the DRBEM.

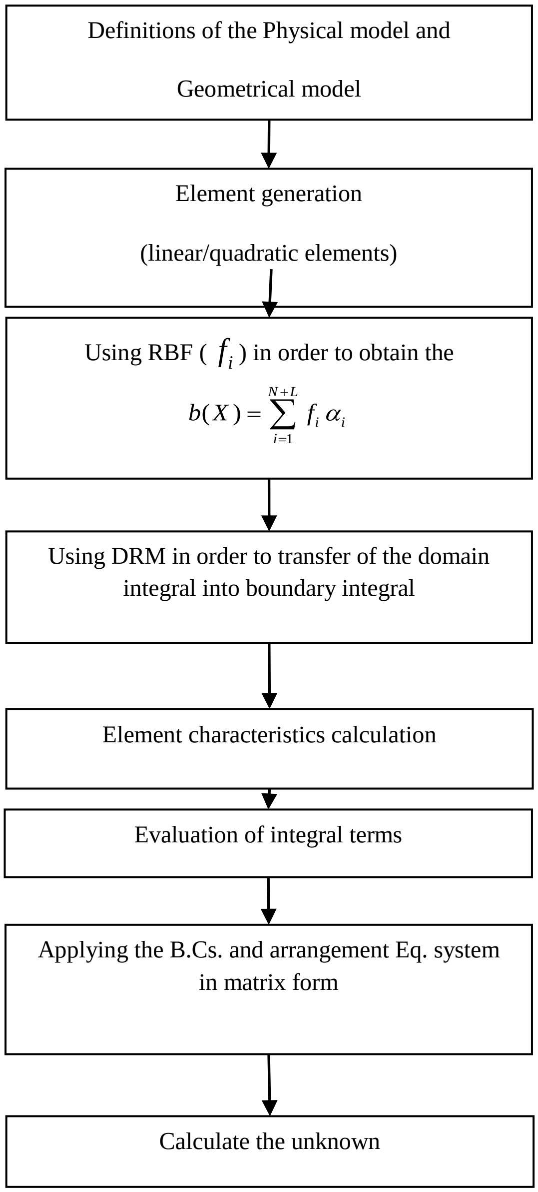

Fig: 4 Flowchart algorithm of the DRBEM

6. Conclusions

The Governing equation of the acoustic problem in three-dimensional potential flow was presented. The mathematical form of this physical problem is found the 3D Helmholtz equation. DRM method was applied to solve this equation. Based on this research, following conclusions can be drawn:

- The acoustic propagates inside the silencer and its sound can be formulated by the three dimensional (3D) Helmholtz equation.
- DRM is the most powerful numerical tool to solve this equation.
- The particular solution of the Helmholtz equation is found from the regular integrating.
- Evaluations of the integral can be found using Gaussian numerical and analytical method.
- Flowchart algorithm of the DRBEM was presented to calculate the unknown of the quantity.
- Special techniques should be focused to determine the analytical method of the integrals.

Numerical examples will be examined in the next research and that will be our future plan.

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