International Journal of Scientific Engineering and Technology (ISSN : 2277-1581)
Volume No.3 Issue No.9, pp : 1213-1216 1 Sep 2014

Solution of a Porous Medium Problem Using Laplace Transform

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Abstract : ***The present paper studies the flow of two immiscible fluids passed in a cylindrical porous medium. The porous medium is partially filled with oil and water is then passed through the medium. Due to the non-mixing nature of the fluids, the perturbations occur. The phenomenon of instabilities in the polyphasic flow through homogeneous porous medium is discussed under the assumption of mean capillary pressure. The resulting governing equation is a linear partial differential equation. The equation gives the saturation of water at any point in the cylinder at any time. In the present paper, the analytical solution has been obtained using the Laplace transform. Also the solution has been analysed graphically.***

Key Words : Laplace transform, immiscible fluids, instabilities, porous medium

Introduction

When a porous medium filled with some fluid is brought into contact with another fluid which preferentially wets the medium, it is observed that there is a spontaneous flow of the wetting fluid into the medium and a counter flow of the resident fluid from the medium. This arises in physical situations involving multiphase flow systems.

The phenomenon of instabilities in polyphasic flow through homogeneous porous medium without capillary pressure was discussed by Scheidegger and Johnson. The behaviour of instabilities in a displacement process through heterogeneous porous medium with capillary pressure was examined by Verma. In the present paper, the phenomenon of instabilities in polyphasic flow through homogeneous porous medium with mean capillary pressure has been discussed. The resulting governing equation is a non-linear partial differential equation.

Formulation of the Problem

We consider a cylindrical mass of porous matrix of length L that is initially saturated with a non-wetting fluid say oil. We assume that the lateral boundaries of the medium as well as one of the cross-sectional faces are impermeable while the only remaining open end is exposed to an adjacent formation of fluid say water which wets the medium preferentially relative to oil. Such circumstances give rise to the phenomenon of linear counter current imbibitions in which there is a spontaneous linear flow of wetting fluid i.e. water into the porous medium and a linear counter flow of the native fluid i.e. oil from the medium. This gives rise to the instabilities at the interface between the two fluids.

For the flow system, the seepage velocity of the wetting and the non- wetting phases are given respectively as

$$v_w = -\frac{k_w}{\gamma_w} K \frac{\partial P_w}{\partial x} \tag{1}$$

$$v_o = \frac{k_o}{\gamma_o} K \frac{\partial P_o}{\partial x} \tag{2}$$

where k_w and k_o are the respective relative permeabilities of water and oil, K is the permeability of the porous medium, P_w and P_o are the pressures and γ_w and γ_o are the viscosities of the water and oil respectively.

Using the mathematical condition for the imbibitions phenomenon $v_w + v_o = 0$, we will get

$$\frac{k_w}{\gamma_w}\frac{\partial P_w}{\partial x} + \frac{k_o}{\gamma_o}\frac{\partial P_o}{\partial x} = 0 \tag{3}$$

From the definition of capillary pressure, we will get

$$P_o = P_c + P_w$$

Differentiating with respect to x, we will get

$$\frac{\partial P_o}{\partial x} = \frac{\partial P_c}{\partial x} + \frac{\partial P_w}{\partial x} \tag{4}$$

Using equation (4) in (3) and simplifying, we will get

$$\frac{\partial P_w}{\partial x} = \frac{-\dfrac{k_o}{v_o}}{\dfrac{k_w}{\gamma_w} + \dfrac{k_o}{\gamma_o}} \quad \frac{\partial P_c}{\partial x} \tag{5}$$

Using equation (5) in equation (1), we will get

$$v_w = K\frac{k_w k_o}{k_w\gamma_o + k_o\gamma_w}\quad\frac{\partial P_c}{\partial x} \tag{6}$$

The equation of continuity for wetting phase is given by

$$P\frac{\partial S_w}{\partial t} + \frac{\partial v_w}{\partial x} = 0 \tag{7}$$

The capillary pressure P_c is a decreasing function of saturation of wetting phase and is related as

$$P = -\beta\ S_w \tag{8}$$

Using equation (8) in equation (6) and taking $\frac{k_w k_o}{k_w\gamma_o + k_o\gamma_w} = \frac{k_o}{\gamma_o}$, we will get

$$v_w = -K\beta\frac{k_w k_o}{k_w\gamma_o + k_o\gamma_w}\frac{\partial^2 S_w}{\partial x^2} \tag{9}$$

Using equation (9) in equation (7), we will get

$$P\frac{\partial S_w}{\partial t} = \frac{\beta K}{\gamma_o}\frac{\partial}{\partial x}\left(k_o\frac{\partial S_w}{\partial x}\right) \tag{10}$$

Since the porous medium is homogeneous, the porosity P and permeability K are constants. Taking $a = \frac{k_o\beta K}{P\gamma_o}$, equation (10) will take the form

$$a\frac{\partial^2 S_w}{\partial x^2} = \frac{\partial S_w}{\partial x} \tag{11}$$

Equation (11) is a linear partial differential equation.

The relevant initial and boundary condition are

$$S_w(X,0)=0,\ \ 0\le X\le L$$
$$S_w(0,t)=\delta(t),\ \ \forall t$$
$$S_w(x\to L,t)=0,\ \ \forall t$$
$$S_w(x\to\infty,t)=0,\ \ \forall t$$

where δ is Direct-delta function.

Multiplying each term of (11) by $e^{-st}\,dt$ and then integrating the resulting equation from 0 to ∞, we will get

$$\int_0^\infty e^{-st}\left(a\frac{\partial^2 S_w}{\partial x^2}\right)dt = \int_0^\infty e^{-st}\frac{\partial S_w}{\partial t}dt$$

$$a\frac{d^2\overline{S_w}}{dx^2} = s\overline{S_w} \tag{12}$$

where $\overline{S_w}(x,s) = \int_0^\infty e^{-st}S_w(x,t)\,dt$ represents the Laplace transform of $S_w(x,t)$

Now equation (12) is an ordinary differential equation with constant coefficients.

Its solution is given by

$$\overline{S_w}(x,s) = c_1 e^{x\sqrt{\frac{s}{a}}} + c_2 e^{-x\sqrt{\frac{s}{a}}} \tag{13}$$

$S_w(x,t)$ is saturation of water at any face x at any time t and a is constant depending on the medium.

The boundary conditions

$$S_w(x,0) = 0\ \ and\ S_w(0,t) = \delta(t)$$

indicate the saturation of water from $x = x\ to\ x = 0$ at time $t = 0\ to\ t = t$

$$L[S_w(x,0)] = \overline{S_w}(x, s=0) = 0$$
$$L[S_w(0,t)] = \overline{S_w}(0,s) = 1$$

Now choose $c_1 = 0$ so that, $\overline{S_w}$ is bounded as $x\to\infty$. From (13), we will get

$$\overline{S_w}(x,s) = c_2 e^{-x\sqrt{\frac{s}{a}}}$$

Using $L[S_w(0,t)] = \overline{S_w}(0,s) = 1$, we will get $c_2 = 1$

Equation (13) will become

$$\overline{S_w}(x,s) = e^{-x\sqrt{\frac{s}{a}}}$$

Taking inverse Laplace transform we will get,

$$S_w(x,t) = \frac{1}{2\sqrt{a\pi}}\frac{x}{t^{\frac{3}{2}}}\exp\left(\frac{-x^2}{4at}\right)$$

Therefore, the saturation of water S_w at any time $t > 0$ and for length $0 < x < L$ is given by

$$S_w(x,t) = \frac{1}{2\sqrt{a\pi}} \frac{x}{t^{\frac{3}{2}}} \exp\left(\frac{-x^2}{4at}\right) \tag{14}$$

Graphical Representation

Using Maple-12, the solution obtained in equation (14) is plotted for different values of x and t.

Here we have assumed the value of a as one. The graphs are shown as below-

$$S_w = \frac{1}{2\sqrt{\pi}} . \frac{x}{t^{\frac{3}{2}}} .\exp\left(-\frac{x^2}{4\ t}\right)$$

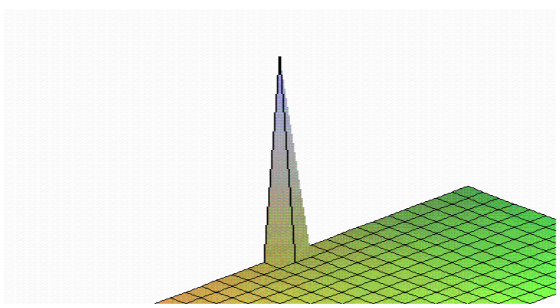
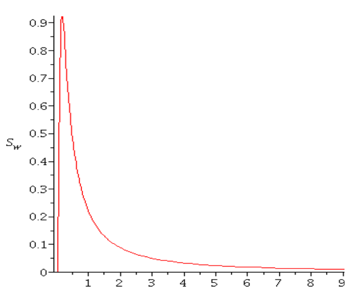


Figure : 1

$$S_w = \frac{1}{2\sqrt{\pi}} . \frac{1}{t^{\frac{3}{2}}} .\exp\left(-\frac{1}{4\ t}\right) \qquad S_w = \frac{1}{2\sqrt{\pi}} . \frac{10}{t^{\frac{3}{2}}} .\exp\left(-\frac{100}{4\ t}\right)$$



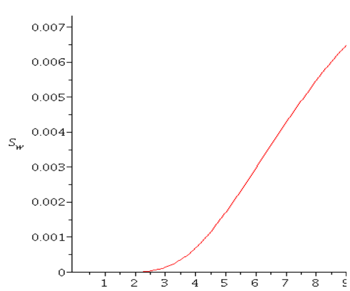
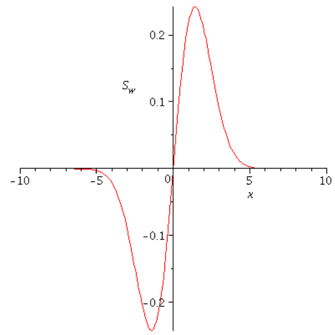


Figure : 2(A) For $x = 1$ Figure : 2(B) For $x = 10$

$$S_w = \frac{1}{2\sqrt{\pi}} .\ x .\exp\left(-\frac{x^2}{4}\right) \qquad S_w = \frac{1}{2\sqrt{\pi}} . \frac{x}{27} .\exp\left(-\frac{x^2}{36}\right)$$



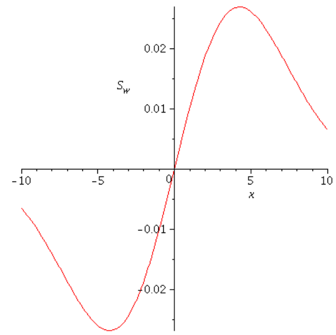
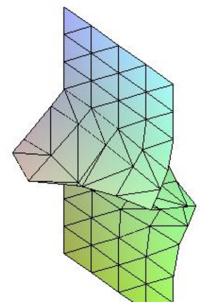


Figure : 3(A) For $t = 1$ Figure : 3(B) For $t = 9$

$$S_w = \frac{1}{2\sqrt{\pi}} . \frac{x}{t^{\frac{3}{2}}} .\exp\left(-\frac{x^2}{4\ t}\right) \qquad S_w = \frac{1}{2\sqrt{\pi}} . \frac{x}{t^{\frac{3}{2}}} .\exp\left(-\frac{x^2}{4\ t}\right)$$



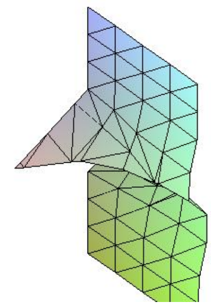


Figure : 4(A) Varying t Between -0.5 *to* 0.5 Figure : 4(B) Varying t Between -0.05 *to* 0.05

Conclusions

The instabilities at the interface between the two immiscible fluids give rise to fingers. These perturbations are shown in figures 1, 4(a) and 4(b). Keeping x constant and with the increase in the value of t, we can see that the effect of exponential term is neglected but because saturation of water is inversely proportional to t, its value decreases with the increase in the value of t. This can be seen in figures 2(a) and 2(b) where it is plotted for two different values of x. Keeping t constant, we can see that the saturation of water occurs as a product of linear function of x and the negative exponential function of x. With the increase in the value of x, the nature of saturation is parabolic and it has the value zero for large values of x. This has been plotted for different values of t and are shown in figures 3(a) and 3(b).

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