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An Analytic Research on Steady Heat Conduction through a Circular Cylinder

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Abstract—The steady heat conduction through a circular cylinder shell is studied theoretically by applying the method of variable separation. The cylinder shell is under conditions of constant properties, constant inner wall temperature, variable outer wall temperature both in length and azimuth. Expressions for temperature distribution and heat flow are developed. The characteristics is discussed under different variations of outer wall temperature.

Keywords—**Circular Cylinder, Steady State, Heat Conduction, Variable Separation, Analytic Solution**

1. Introduction

Circular tubes have long been widely used in many industries of heat and mass transfer because of its high structural strength, easy handling and convenient operation. Up to now there have been extensive studies on heat transfer through circular tubes in serious conditions such as periodic[1], layered materials[2,3], convective coupled[4], or innovative means[5,6]. Circular tubes have two chief objectives in heat transfer engineering. One is to serve as heat transfer section surfaces to realize heat energy exchange between two fluid medium flowing inside and outside the tubes respectively. In this case, the main task is to enhance the heat transfer capacity of the tube section as much as possible. The other objective of tubes is to be apparatus of transporting some wanted fluid medium from one place to another. In this condition, it is desired to eliminate leakage of the medium out and minimize heat transfer between the tube and the surroundings in order to maintain the medium properties constant.

In the process of heat transfer through circular tubes, temperature in the tubes varies generally not only along the tube length but also along the radius and the azimuth as well. So the heat transfer occurring in tubes in almost all cases are of problems of unsteady and three dimensional. However, many cases could be simplified to be steady and two or even one dimensional problems according to concrete situations. Indeed, the steady heat transfer models neglecting temperature variations in the tube azimuth and length have simple and straight forward solutions, and have been written down in many heat transfer books[7,8,9] long before. In many situations however, such negligence might causes severe loss. For instance, in the heat transfer of fluid flowing perpendicular to a circular tube, the temperature of the inner tube wall at each cross section might be considered to be invariant in azimuth. Yet for the outer tube wall, the temperature might changes remarkably not only along the tube length, but also obvious along the azimuth as well because of the heat transfer between the outer wall and the outside surroundings. In this situation, above mentioned models in present text or handbooks could result in large errors in predicting the tube wall temperature and in designing new innovative compact heat exchangers[10] .

In this study, a theoretical investigation is carried out on the heat conduction through a circular cylinder shell in steady state in which the temperature varies only along the tube length on the inner tube wall but changes both along the tube azimuth and length on the outer tube wall. The heat conduction in the tube length or axial direction resulted from the temperature gradient in the tube length is taken to be the effect of an inner heat source. By using variable separation, expressions for temperature distribution in the tube shell and heat conduction rate through the tube are obtained. Effects of the temperature variations in azimuth of the outer wall are discussed.

2. Formulation

2.1 Problem

Consider a problem of steady heat conduction through a circular cylinder shell shown in figure 1 with Cartesian and polar coordinate systems as well. The inner and outer radius of the tube wall are r_1 and r_2 respectively. The tube has a length of L and a constant heat conductivity of λ. The temperature of an arbitrary point r,θ, z within the shell is denoted as $t(r,\theta,z)$. The inner wall temperature keeps constant in azimuth but varies along the tube length and is denoted as $t_1(z)$. The outer wall temperature is symmetry about x coordinate, varying along the length and azimuth and is denoted to be $t_2(\theta,z)$. The temperature of the left and right boundary walls are denoted to be $t_3(r,\theta)$ and $t_4(r,\theta)$ respectively. Under such conditions, we treat the heat transfer in tube length to be resulted from an inner heat source of q_v in the tube wall. Then the governing energy equation with its boundary conditions are as follows.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right)+\frac{\partial^2 t}{r^2\,\partial\theta^2}+\frac{\partial^2 t}{\partial z^2}+\frac{q_v}{\lambda}=0 \qquad (1)$$

$$r=r_1\,,\quad t=t_1(z) \qquad (2)$$

$$r=r_2\,,\quad t=t_2(\theta, z) \qquad (3)$$

$$z=0,\quad t=t_3(r,\theta) \qquad (4)$$

$$z=L,\quad t=t_4(r,\theta) \qquad (5)$$

$$\partial t(r,\theta=0,z)/\partial\theta=\partial t(r,\theta=\pi,z)/\partial\theta\,. \qquad (6)$$

2.2 Simplification

2.2.1 Treatment of Temperature

As mentioned above, the temperature of the inner wall of the tube is assumed being constant along azimuth but varies along length. Still more, it is further assumed that the axial temperature

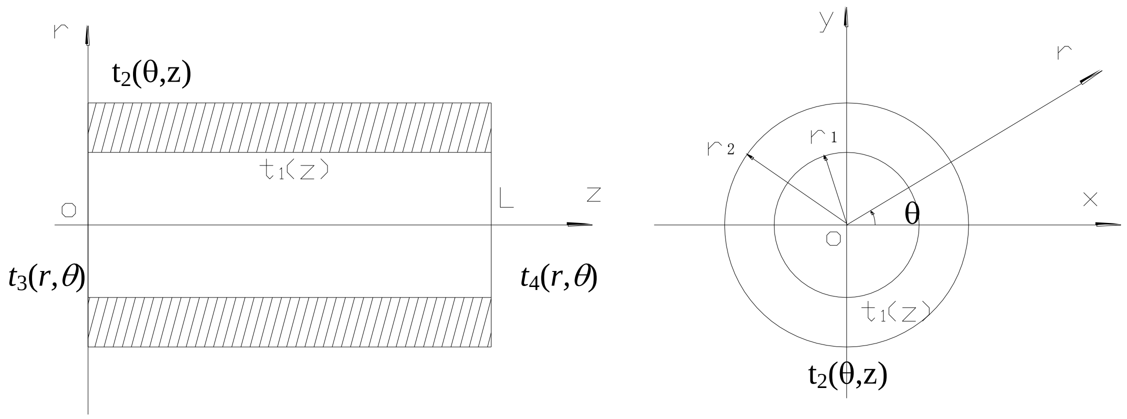

Fig. 1 Circular Cylinder Heat Conduction

variation of the inner wall represents meanwhile the temperature variation of every point in the cross section at the same axial location as the inner wall. Under such conditions, the temperature at point r,θ, z within the tube shell could be written as

$$t(r,\theta,z)= t_1(z)+T(r,\theta)\ . \tag{7}$$

Substituting Eq.(7) into Eqs.(1),(2) and (3) yields

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{r^2\,\partial\theta^2}+\frac{\partial^2 t_1}{\partial z^2}+\frac{q_v}{\lambda}=0 \tag{8}$$

$$r=r_1,\ T=0 \tag{9}$$

$$r=r_2,\ T=t_2(\theta,z)-t_1(z)=f(\theta)\ . \tag{10}$$

By applying Eq.(7), Eq.(5) - Eq.(4) gives

$$t_4(r,\theta)-t_3(r,\theta))=t_1(L)-t_1(0)\ . \tag{11}$$

Equation (10) implies that $t_1(z)$ and $t_2(\theta,z)$ have same change in axial direction. Equation (11) indicates that with negligence of temperature difference in both radius and azimuth at the tube cross section, the temperature change between left and right boundary surfaces could be determined by the temperature change along the tube length.

2.2.2 Inner heat source

The component of heat conduction through the tube length is considered to be the result of an inner heat source, by which the overall heat generation rate is

$$Q_z=\frac{\lambda\pi(r_2^2-r_1^2)(t_{4m}-t_{3m})}{L}=\frac{\lambda\pi(r_2^2-r_1^2)(t_1(L)-t_1(0))}{L}\ , \tag{12}$$

Where t_{3m} and t_{4m} denote the mean temperature of the left and right boundary walls respectively. Then the heat generation rate of the inner heat source of unit volume of the solid tube shell becomes

$$q_v=\frac{\lambda\pi(t_1(L)-t_1(0))}{L^2}\ . \tag{13}$$

3. Solution

3.1Temperature t_1

Recalling the definition of Eq. (7), decompose Eq.(8) and let

$$\frac{\partial^2 t_1}{\partial z^2}+\frac{q_v}{\lambda}=0\ . \tag{14}$$

With the conditions $z=0, t_1(z)=t_1(0)$ and $z=L, t_1(z)=t_1(L)$, the solution to Eq. (14) could be

$$t_1(z)=t_1(0)+\frac{t_1(L)-t_1(0)}{L}z+\frac{1}{2}\frac{q_v}{\lambda}(Lz-z^2) \tag{15}$$

3.2 Temperature *T*

With the method of variable separation, set

$$T(r,\theta)=R(r)\Phi(\theta). \tag{16}$$

Substituting Eq. (16) into Eq. (8) with accounting for Eq. (14), one obtains

$$\frac{\partial^2 R}{\partial r^2}\Phi+\frac{1}{r}\frac{\partial R}{\partial r}\Phi+\frac{\partial^2\Phi}{r^2\,\partial\theta^2}R=0\ , \tag{17}$$

divided by Eq. (16), Eq. (17) is deduced to

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}=-\frac{\partial^2\Phi}{\Phi\,\partial\theta^2}=n^2\ . \tag{18}$$

Where n is an arbitrary constant number. After some rearrangement, two ordinary differential equations are deduced to be

$$r^2\frac{d^2R}{dr^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}-n^2R=0 \tag{19}$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\theta^2}+n^2\Phi=0\ . \tag{20}$$

The boundary conditions to Eq. (20) are

$$\theta=0,\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}=0\quad \theta=\pi,\frac{\mathrm{d}\Phi}{\mathrm{d}\theta}=0\ \text{and}\ \Phi(\theta+2k\pi)=\Phi(\theta)\ , \tag{21}$$

where k is an arbitrary integer number. The third expression in Eq. (21) means that what ever the solutions might be, it would be the same point and thereafter the temperature would be the same.

Deducing from Eqs. (20) and (21) with n=0，one gets

$$\Phi_0=c_{00}\ , \tag{22}$$

where c_{00} is an unknown constant. When n=1,2,3……， we have

$$\Phi_n = c_{0n} \cos(n\theta). \tag{23}$$

Solving Eq. (19) with n=0 yields

$$R_0 = c_{10} + c_{11} \ln r. \tag{24}$$

When n=1,2,3……, there gives

$$R_n = c_{2n} r^n + c_{3n} r^{-n}. \tag{25}$$

From boundary condition Eq. (9), one get R=0 at $r=r_1$. Substituting it with some rearrangement, Eqs. (24) and (25) become respectively as

$$R_0 = c_{10} - c_{10} \ln r / \ln r_1 \tag{26}$$

$$R_n = c_{2n} r^n - c_{2n} r_1^{2n} r^{-n}, \tag{27}$$

then

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta) = \sum_{0}^{\infty} R_n(r)\Phi_n(\theta)$$

$$= c_0(1 - \ln r / \ln r_1) + \sum_{n=1}^{\infty} c_n (r^n - r_1^{2n} r^{-n}) \cos(n\theta), \tag{28}$$

where $c_0=c_{00}c_{10}$, $c_n=c_{0n}c_{2n}$. All of the constants c are arbitrary and need to be evaluated with the other boundary condition Eq. (10). After substitution Eq. (10) into Eq. (28), one arrives at

$$T(r_2,\theta) = c_0(1 - \ln r_2 / \ln r_1) + \sum_{n=1}^{\infty} c_n (r_2^n - r_1^{2n} r_2^{-n}) \cos(n\theta) = f(\theta). \tag{29}$$

Then

$$c_0 = \frac{1}{\pi} \int_0^{\pi} \frac{f(\theta)}{1 - \ln r_2 / \ln r_1} d\theta \tag{30}$$

$$c_n = \frac{2}{\pi} \int_0^{\pi} \frac{f(\theta)}{r_2^n - r_1^{2n} r_2^{-n}} \cos(n\theta) d\theta \quad n=1,2,3\ldots\ldots \tag{31}$$

By applying Eqs. (15) and (28) into Eq. (7), the temperature of every point within the circular cylinder shell could be calculated.

3.3 Rate of Heat Conduction

In present problem, the rate of heat conduction through a circular cylinder shell consists of the heat conducted between the left and right boundary walls and the heat conducted between the inner and outer walls. The former component is dependent upon the overall slope of the temperature variation in the axial direction and could be evaluated by Eq. (12). The latter component, i.e. the heat conduction in the radial direction through the inner and outer walls is

$$Q_r = -2\lambda \int_0^{\pi} \frac{\partial t}{\partial r} r d\theta = -2\lambda \int_0^{\pi} \frac{\partial T}{\partial r} r d\theta$$

$$= 2\pi\lambda L (c_0 / \ln r_1). \tag{32}$$

4. Discussion

In practical cases, when the temperature of the inner wall, outer wall, left and right walls are given, c_0 and c_n could then be calculated with Eqs. (30) and (31). The temperature of every point within the circular cylinder tube could be evaluated by using Eqs. (13), (15) and (28) into (7). The heat conduction rate in the tube length could be calculated by Eq. (12), the heat conduction rate in radius direction could be calculated by Eq. (32). When the temperature of the inner and outer walls change very slightly along the azimuth so that could be ignored, only the change along the radius is apparent, then $t_2(\theta,z)$ - $t_1(z)=f(\theta)$, becomes a constant and is denoted as $f(\theta)=\Delta t$, one gets

$$c_0 = \frac{\Delta t}{1 - \ln r_2 / \ln r_1} \text{ and } c_n = 0 \tag{33}$$

$$Q_r = 2\pi\lambda L \frac{\Delta t}{\ln(r_1/r_2)}. \tag{34}$$

This result reduces to that in general textbooks about one-dimensional heat conduction through circular cylinder in radial direction.

If the inner wall temperature keeps invariant while the outer wall temperature changes lineally along azimuth and expressed as

$$t_2(\theta,z) - t_1(z) = f(\theta) = \Delta t_0 + \varepsilon\theta.$$

Here ε is a constant coefficient. Then it is obtained as

$$c_0 = \frac{\Delta t_0}{1 - \ln r_2 / \ln r_1} + \frac{1}{2} \frac{\varepsilon\pi}{1 - \ln r_2 / \ln r_1} \tag{35}$$

$$c_n = \frac{2\varepsilon}{\pi n^2} \frac{(-1)^n - 1}{r_2^n - r_1^{2n} r_2^{-n}} \tag{36}$$

$$Q_r = 2\pi\lambda L \frac{\Delta t_0 + 1/2\varepsilon\pi}{\ln(r_1/r_2)}. \tag{37}$$

Thirdly, when the outer wall temperature changes non-lineally along azimuth while the inner wall temperature still being invariant, take the temperature difference between the inner wall and outer wall, for example written as

$$t_2(\theta,z) - t_1(z) = f(\theta) = \Delta t_0 + \beta\theta^2.$$

Here β is also a constant coefficient. Then we have

$$c_0 = \frac{\Delta t_0}{1 - \ln r_2 / \ln r_1} + \frac{1}{3} \frac{\beta\pi^2}{1 - \ln r_2 / \ln r_1} \tag{38}$$

$$c_n = \frac{4\beta}{n^2} \frac{(-1)^n}{r_2^n - r_1^{2n} r_2^{\,n}} \tag{39}$$

$$Q_r = 2\pi\lambda L \frac{\Delta t_0 + 1/3\beta\pi^2}{\ln(r_1/r_2)}. \tag{40}$$

At these results, one can concludes that when the outer wall temperature varies while that of the inner wall keeps constant along the tube azimuth, temperature of every point within the solid cylinder will definitely differ from that of the radial heat conduction with the inner and outer wall temperatures both invariant along the azimuth. As for the radial heat flux, there also seems apparently to be of significant differences. In order to make this question clear, further analysis is needed.

The question to be considered is that, while the inner wall temperature is azimuthal constant, the outer wall temperature varies lineally or non-lineally in azimuth. No matter how the outer wall temperature changes, it has an equivalent mean value along the azimuth. That is the outer wall temperature varies along the azimuth equivalently above or down an same value. For convenience, let the difference between the invariant inner wall temperature and the mean temperature of the outer wall be Δt.

When the outer wall temperature changes linearly, in accordance with above presumption, we have such relation

$$\int_0^{\pi}(\Delta t_0 + \varepsilon\theta)\mathrm{d}\theta = \int_0^{\pi}\Delta t\mathrm{d}\theta .$$

And then

$$\Delta t_0 + 1/2\varepsilon\pi = \Delta t. \quad (41)$$

When the outer wall temperature changes quadratically in azimuth, we have

$$\int_0^{\pi}(\Delta t_0 + \beta\theta^2)\mathrm{d}\theta = \int_0^{\pi}\Delta t\mathrm{d}\theta .$$

Then it is got

$$\Delta t_0 + 1/3\beta\pi^2 = \Delta t. \quad (42)$$

Recalling Eqs. (37) and (40), we see that it is obvious that the rate of radial heat conduction remains the same for the outer wall temperature changing in the azimuth linearly or quadratically as long as their mean temperature being the same.

From this point, in the conditions of invariant inner wall temperature along azimuth, we may deduce that no matter how the outer wall temperature varies along the azimuth, so long as the mean temperature keeps the same, the radial heat flux conducted though a circular tube remains the same. As for the temperature within the tube shell, present approach could play an appropriate role. $c_0, c_1, c_2, \ldots c_n$ should be calculated by Eqs. (30) and (31) with related inner and outer wall temperature substituted, the number of constants should be chosen according to practical situation. After all, temperature could be evaluated by using Eqs. (7), (13),(15) and (28) of this study.

Under present treatment, heat conduction in tube length and tube radius don't interfere each other, but exert additional effects in temperature.

5. Conclusion

In this study the steady heat conduction in a circular cylinder both through radial and axial directions is investigated theoretically for the conditions in which the temperature of the inner wall keeps constant but that of the outer wall varies along the azimuth, apart from this, both the inner and outer wall temperature vary equivalently in axial direction. The method of variable separation is used in which the heat conduction in axial direction is treated as the effect of an inner heat source. Taking the inner wall temperature as starting point, the partial energy equation is separated into two sets of ordinary equations, and solved respectively. The result illustrates that under present conditions, the temperature distribution within the circular cylinder is determined by the variation of the outer wall temperature both in azimuthal and axial direction. It is indicated that when the outer wall temperature changes along the azimuth around a mean value, the radial heat conduction could be calculated by the means provided in present text or handbooks with the mean outer wall temperature used in the temperature difference.

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