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SOME SOLUTIONS FOR AN EQUATION OF ORDER OF 4p

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Abstract- We obtain all solutions which depend only on r for a singular partial differential equation of order 4p. Here, the operator includes Laplacian and GASPT (Generalized Axially Symmetric Potential Theory) operator.

Keywords- Singular equation, Solution of type r^m, Laplacian, GASPT operator.

1. INTRODUCTION

This paper consists of solutions of type r^m for the linear partial differential equation of order of 4p

$$L^p(u) = 0 \tag{1.1}$$

where p is a positive integer and the singular operator L is defined by

$$L = r^2 \sum_{i=1}^{n} \left\{ \frac{\lambda a_i^6 r^2}{\left(x_i - x_i^0\right)^2} \frac{\partial^4}{\partial x_i^4} + \left[\frac{\mu a_i^4}{x_i - x_i^0} - \frac{\lambda a_i^6 r^2}{\left(x_i - x_i^0\right)^3} \right] \frac{\partial^3}{\partial x_i^3} \right\} + \sum_{i=1}^{n} \left(a_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i}{x_i - x_i^0} \frac{\partial}{\partial x_i} \right) + \frac{\gamma}{r^2} \tag{1.2}$$

The iterated operators L^p are defined by the relations

$$L^{k+1}(u) = L\left[L^k(u)\right]; \quad k = 1,\ldots,p-1$$

In (1.2) x_i^0, $a_i \neq 0$ (i=1,...,n) are real constants and λ, μ, γ, α_i (i=1,...,n) are any real parameters and r is given by

$$r = \left[\sum_{i=1}^{n}\left(\frac{x_i - x_i^0}{a_i}\right)^2\right]^{1/2}, \qquad r > 0 \tag{1.3}$$

Equation (1.1) includes some well-known classical equations such as the Laplace equation, GASPT equation and their iterated forms. Many authors studied these equations in solving some physical problems [1-5].

2. SOLUTIONS OF TYPE r^m

Firstly, we will give the following lemma.

Lemma 2.1 Let p be a positive integer and m be a real or complex parameter. Then

$$L^p(r^m) = \prod_{k=0}^{p-1}\Phi(m-2k)r^{m-2p} \tag{2.1}$$

where $\Phi(m)$ is a fourth degree polynomial given by

$$\begin{aligned}\Phi(m) &= \lambda m^4 + (5\lambda n - 12\lambda + \mu)m^3 + (44\lambda - 30\lambda n + 3n\mu - 6\mu + 1)m^2 \\ &\quad + (40\lambda n - 48\lambda - 6n\mu + 8\mu + n - 2 + \rho)m + \gamma\end{aligned} \tag{2.2}$$

with $\rho = \sum_{i=1}^{n}\frac{\alpha_i}{a_i^2}$.

Proof. From the definitions of L and r for any real or complex parameter m, we have

$$\begin{aligned}L(r^m) &= \left[\lambda m^4 + (5\lambda n - 12\lambda + \mu)m^3 + (44\lambda - 30\lambda n + 3n\mu - 6\mu + 1)m^2\right. \\ &\quad \left. + (40\lambda n - 48\lambda - 6n\mu + 8\mu + n - 2 + \rho)m + \gamma\right]r^{m-2} \\ &= \Phi(m)r^{m-2}\end{aligned} \tag{2.3}$$

Applying the operator L consecutively p-1 times on both sides of (2.3), we obtain the formula (2.1).
Now consider the formula (2.1) and write the algebraic polynomial equation

$$\prod_{k=0}^{p-1}\Phi(m-2k) = 0 \tag{2.4}$$

which is degree of 4p. The number of real or complex roots of the equation (2.4) is 4p for $\lambda \neq 0$.

Now using Lemma 2.1, we can prove the following theorem.

Theorem 2.1 Let the algebraic polynomial equation (2.4) have distinct roots $c_1, c_2, \ldots, c_M$ each having multiplicity $\xi_1, \xi_2, \ldots, \xi_M$, respectively, and all of the roots be real. Then the solution of type r^m for the equation (1.1) is given by the formula

$$u(r) = \sum_{j=1}^{M} \sum_{\upsilon=0}^{\xi_j - 1} A_{j\upsilon} r^{c_j} (\ln r)^{\upsilon} \tag{2.5}$$

where $A_{j\upsilon}$ are arbitrary constants.

Proof. According to the hypothesis we can write the algebraic equation (2.4) as

$$\lambda^p \prod_{j=1}^{M} \left(m - c_j\right)^{\xi_j} = 0$$

where $\sum_{j=1}^{M} \xi_j = 4p$ is degree of (2.4). Therefore, the formula (2.1) can be written as

$$L^p\left(r^m\right) = \lambda^p \prod_{J=1}^{M} \left(m - c_j\right)^{\xi_j} r^{m-2p} \tag{2.6}$$

On the other hand, truth of the following equalities can be showed easily by induction.

$$\frac{\partial^k}{\partial m^k}\left[L^p\left(r^m\right)\right] = L^p\left(\frac{\partial^k r^m}{\partial m^k}\right) = L^p\left[r^m (\ln r)^k\right], \; k \in N \tag{2.7}$$

Now again consider (2.6). It is obvious that the right-hand side of (2.6) has the factors $\left(m - c_j\right)^{\xi_j}$; j=1,...,M which vanish for $m = c_j$; j=1,...,M together with its derivatives with respect to m

$$\frac{d^{\upsilon}}{dm^{\upsilon}}\left(m - c_j\right)^{\xi_j}; \quad \upsilon = 1, \ldots, \xi_j - 1, \; j = 1, \ldots, M.$$

Thus, the functions r^{c_j} for j=1,...,M and from (2.7) each of the functions

$$\left.\frac{\partial^{\upsilon} r^m}{\partial m^{\upsilon}}\right|_{m=c_j} = r^{c_j} (\ln r)^{\upsilon}; \quad \upsilon = 1, \ldots, \xi_j - 1, \; j = 1, \ldots, M$$

satisfy the equation (1.1). Since the given equation is linear, by the superposition principle the sum

$$u(r)=\sum_{j=1}^{M}\sum_{\upsilon=0}^{\xi_j-1}A_{j\upsilon}r^{c_j}(\ln r)^{\upsilon}$$

also satisfies (1.1). Thus, the theorem is proved.

Theorem 2.2 Let the algebraic polynomial equation (2.4) have distinct roots $\alpha_1 \pm i\beta_1$, $\alpha_2 \pm i\beta_2;\ldots,\alpha_N \pm i\beta_N$ each having multiplicity τ_1, $\tau_2,\ldots,\tau_N$, respectively, and all of the roots be complex. Then the solution of type r^m for equation (1.1) is given by the formula

$$u(r)=\sum_{s=1}^{N}\sum_{q=0}^{\tau_s-1}r^{\alpha_s}(\ln r)^q\left[B_{sq}\cos(\beta_s\ln r)+C_{sq}\sin(\beta_s\ln r)\right] \tag{2.8}$$

where B_{sq} ve C_{sq} are arbitrary constants.

Proof. Similar to Theorem 2.1, we can write (2.1) as

$$L^p(r^m)=\lambda^p\prod_{s=1}^{N}\left(m^2-2\alpha_s m+\alpha_s^2+\beta_s^2\right)^{\tau_s}r^{m-2p} \tag{2.9}$$

where $2\sum_{s=1}^{N}\tau_s=4p$. The factors

$$\left(m^2-2\alpha_s m+\alpha_s^2+\beta_s^2\right)^{\tau_s}=\left[m-(\alpha_s+i\beta_s)\right]^{\tau_s}\left[m-(\alpha_s-i\beta_s)\right]^{\tau_s};\ s=1,\ldots,N$$

which are on the right-hand side of (2.9) and the following derivatives of these factors

$$\frac{d^q}{dm^q}\left[m-(\alpha_s\pm i\beta_s)\right]^{\tau_s};\ q=1,\ldots,\tau_s-1,\quad s=1,\ldots,N$$

are zero for $m=\alpha_s\pm i\beta_s$. Therefore, for s=1 ,...,N each of the functions $r^{\alpha_s\pm i\beta_s}$ and , from (2.7) and the following expression

$$r^{\alpha_s\pm i\beta_s}=r^{\alpha_s}r^{\pm i\beta_s}=r^{\alpha_s}e^{\pm i\beta_s\ln r}=r^{\alpha_s}\left[\cos(\beta_s\ln r)\pm i\sin(\beta_s\ln r)\right]$$

for $q=1,\ldots,\tau_s-1$, $s=1,\ldots,N$ each of the functions

$$\left.\frac{\partial^q r^m}{\partial m^q}\right|_{m=\alpha_s \pm i\beta_s} = r^{\alpha_s \pm i\beta_s}(\ln r)^q = r^{\alpha_s}(\ln r)^q\left[\cos\beta_s(\ln r) \pm i\sin\beta_s(\ln r)\right]$$

satisfy the equation (1.1). Since the given equation is linear, by the superposition principle the sum (2.8) also satisfies (1.1).

Now, we can give the following theorem which is a result of the Theorem 2.1 and Theorem 2.2. Proof of the theorem is similar to the previous theorems.

Theorem 2.3 Let the algebraic polynomial equation (2.4) have distinct real roots $c_1, c_2, \ldots, c_M$, each having multiplicity $\xi_1, \xi_2, \ldots, \xi_M$, respectively, and distinct complex roots $\alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2; \ldots, \alpha_N \pm i\beta_N$, each having multiplicity $\tau_1, \tau_2, \ldots, \tau_N$, respectively. Then the solutions of type r^m for equation (1.1) is given by the formula

$$u(r) = \sum_{j=1}^{M}\sum_{\upsilon=0}^{\xi_j - 1} A_{j\upsilon} r^{c_j}(\ln r)^{\upsilon} + \sum_{s=1}^{N}\sum_{q=0}^{\tau_s - 1} r^{\alpha_s}(\ln r)^q\left[B_{sq}\cos(\beta_s \ln r) + C_{sq}\sin(\beta_s \ln r)\right] \qquad (2.10)$$

where $A_{j\upsilon}$, B_{sq} ve C_{sq} are arbitrary constants.

Example 2.1 Consider the following operator L

$$L = r^4\left[\frac{1}{(x+1)^2}\frac{\partial^4}{\partial x^4} + \frac{64}{(y-3)^2}\frac{\partial^4}{\partial y^4} - \frac{1}{(x+1)^3}\frac{\partial^3}{\partial x^3} - \frac{64}{(y-3)^3}\frac{\partial^3}{\partial y^3}\right.$$
$$\left. + \frac{\partial^2}{\partial x^2} + 4\frac{\partial^2}{\partial y^2} + \frac{2}{x+1}\frac{\partial}{\partial x} - \frac{8}{y-3}\frac{\partial}{\partial y} - \frac{16}{r^2}\right] \qquad (2.11)$$

where $r^2 = (x+1)^2 + \frac{(y-3)^2}{4}$.

Now, we obtain solution of type r^m for equation $L^2(u) = 0$. If we substitute $\lambda = 1$, $\mu = 0$, $a_1 = 1$, $a_2 = -2$, $\alpha_1 = 2$, $\alpha_2 = -8$, $\gamma = -16$, $x_1^0 = -1$, $x_2^0 = 3$ which are the parameter values in the operator (2.11) in (2.2), we then find

$$\phi(m) = m^4 - 2m^3 - 15m^2 + 32m - 16$$
$$= (m+4)(m-1)^2(m-4)$$

and we can write from (2.1)

$$L^2(r^m) = \phi(m)\phi(m-2)r^{m-4}$$
$$= \{(m+4)(m+2)(m-1)^2(m-3)^2(m-4)(m-6)\}r^{m-4}$$

Therefore, algebraic polynomial equation is

$$(m+4)(m+2)(m-1)^2(m-3)^2(m-4)(m-6) = 0$$

From this equation we obtain $c_1 = 6$, $c_2 = 4$, $c_3 = 3$, $c_4 = 1$, $c_5 = -2$, $c_6 = -4$, $\xi_1 = 1$, $\xi_2 = 1$, $\xi_3 = 2$, $\xi_4 = 2$, $\xi_5 = 1$, $\xi_6 = 1$. If these values are substituted in (2.5) we then have

$$u(r) = \sum_{j=1}^{6}\sum_{\upsilon=0}^{\xi_j-1} A_{j\upsilon} r^{c_j}(\ln r)^{\upsilon} = A_{10}r^6 + A_{20}r^4 + A_{30}r^3 + A_{31}r^3(\ln r)$$
$$+ A_{40}r + A_{41}r(\ln r) + A_{50}r^{-2} + A_{60}r^{-4}$$

solution of type r^m for $L^2(u) = 0$.

3. SOLUTIONS OF TYPE u = u(r)

In this section, we will show that all solutions which depend only on r for the equation (1.1) can be expressed by formula (2.10).

Theorem 3.1 All solutions of type $u = u(r)$ for equation (1.1) can be expressed by the formula (2.10).

Proof. Consider the operator (1.2). Applying the operator L to the function $u = u(r)$, we obtain

$$L(u) = \lambda r^2 \frac{d^4u}{dr^4} + (5\lambda n - 6\lambda + \mu)r\frac{d^3u}{dr^3} + (15\lambda - 15\lambda n + 3\mu n - 3\mu + 1)\frac{d^2u}{dr^2}$$
$$+ (15\lambda n - 15\lambda + 3\mu - 3\mu n + n - 1 + \rho)r^{-1}\frac{du}{dr} + \frac{\gamma u}{r^2} \tag{3.1}$$

Since the above operator is an Euler type operator, if we set $r = e^t$ and $D = \frac{d}{dt}$, then we have

$$L(u) = L(D)u = e^{-2t}\big[\lambda D^4 + (5\lambda n - 12\lambda + \mu)D^3 + (44\lambda - 30\lambda n + 3\mu n - 6\mu + 1)D^2$$
$$+ (40\lambda n - 48\lambda + 8\mu - 6\mu n + n - 2 + \rho)D + \gamma\big]u$$
$$= e^{-2t}\Phi(D)u \tag{3.2}$$

Applying the operator L on both sides of (3.2), we find

$$L^2(u) = L^2(D)u = L(D)\{e^{-2t}\Phi(D)u\} = e^{-2t}\Phi(D)\{e^{-2t}\Phi(D)u\} \quad (3.3)$$

From ordinary differential equations, we know that, for any polynomials of the operator D with constant coefficients G and H and for a constant α, the following relation is valid [1].

$$G(D)\{e^{-\alpha t}H(D)u\} = e^{-\alpha t}G(D-\alpha)H(D)u$$

Considering this property, (3.3) can be written as

$$\begin{aligned} L^2(u) = L^2(D)u &= e^{-2t}\Phi(D)\{e^{-2t}\Phi(D)u\} \\ &= e^{-4t}\Phi(D-2)\Phi(D)u \end{aligned} \quad (3.4)$$

We remark that the product of $\Phi(D-2)$ and $\Phi(D)$ is commutative. Applying the operator L repeatedly p-2 times on both sides of (3.4), then we obtain

$$L^p(u) = e^{-2pt}\prod_{k=0}^{p-1}\Phi(D-2k)u \quad (3.5)$$

Equating the expression (3.5) to zero, we obtain an ordinary differential equation with constant coefficients and of order 4p. The indical equation for this equation is

$$\prod_{k=0}^{p-1}\Phi(m-2k) = 0$$

This was obtained previously on the right-hand side of (2.1). It is obvious that the corresponding solution for this equation is given by (2.10).

We note that, if we substitute $u = r^m$ in (3.5) then, by considering $r^m = e^{mt}$ and $e^{-2pt} = r^{-2p}$ and $\prod_{k=0}^{p-1}\Phi(D-2k)e^{mt} = e^{mt}\prod_{k=0}^{p-1}\Phi(m-2k)$, we see that (3.5) reduces to (2.1).

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