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GROUP CLASSIFICATION AND SOME SIMILARITY SOLUTIONS FOR A NONLINEAR FILTRATION EQUATION

Mehmet Pakdemirli
Department of Mechanical Engineering
Celal Bayar University, 45140
Muradiye, Manisa, Turkey
mpak@bayar.edu.tr

Abstract- A nonlinear filtration equation in which the filter coefficient is an arbitrary function of the specific deposit is considered. Lie Group theory is applied to the coupled system of partial differential equations. Group classification is performed with respect to the arbitrary filter coefficient. Some similarity solutions are constructed using the symmetries.

Key Words- Filtration equation, Lie Group Theory, Group Classification, Similarity Solutions

1.INTRODUCTION

The filtration process is described by coupled partial differential equations [1]

$$\frac{\partial c}{\partial x} + \lambda(\sigma)\, c = 0, \qquad \frac{\partial \sigma}{\partial t} + v\frac{\partial c}{\partial x} = 0 \tag{1}$$

where x is the depth, t is the time, c(x,t) is the concentration, λ is the filter coefficient, $\sigma(x,t)$ is the specific deposit and v is the approach velocity, a constant. The filter coefficient is taken to be an arbitrary function of the deposit.

In search of exact solutions, Lie Group theory will be applied to the partial differential system. Symmetries of the differential equations will be calculated. It will be shown that for some specific forms of the filter coefficient, symmetries will increase. Hence a group classification of the problem will be presented. Using the symmetries, some similarity solutions will be constructed for the problem.

2. GROUP CLASSIFICATION

The infinitesimal generator for the problem is

$$\mathbf{X} = \xi_1(x,t,c,\sigma)\frac{\partial}{\partial x} + \xi_2(x,t,c,\sigma)\frac{\partial}{\partial t} + \eta_1(x,t,c,\sigma)\frac{\partial}{\partial c} + \eta_2(x,t,c,\sigma)\frac{\partial}{\partial \sigma} \tag{2}$$

Writing the equations in terms of higher order variables, prolonging the generator to these variables and requiring the invariance conditions for both equations, one finally obtains the following infinitesimals (See Bluman and Kumei [2] for details of the procedure)

$$\xi_1 = a, \quad \xi_2 = b(t), \quad \eta_1 = -b'(t)c, \quad \eta_2 = 0 \tag{3}$$

The above results are valid for an arbitrary function of the filter coefficient and hence are the principal Lie algebra for the problem. The algebra extends for some specific forms of the filter function which are listed below

i) $\lambda = \lambda_0(\sigma + \alpha)$

$$\xi_1 = a(x), \quad \xi_2 = b(t), \quad \eta_1 = -b'(t)c, \quad \eta_2 = -a'(x)(\sigma + \alpha) \tag{4}$$

ii) $\lambda = \lambda_0$

$$\xi_1 = a, \quad \xi_2 = b(t), \quad \eta_1 = [d - b'(t)]c + h(x,t), \quad \eta_2 = d\sigma + g(x,t) \tag{5}$$

where functions h(x,t) and g(x,t) satisfy

$$\frac{\partial h}{\partial x} + \lambda_0 h = 0, \qquad \frac{\partial g}{\partial t} + v\frac{\partial h}{\partial x} = 0$$

iii) $\lambda = \lambda_0(k\,\sigma + \alpha)^{1/k}$

$$\xi_1 = ax + d, \quad \xi_2 = b(t), \quad \eta_1 = [(1-k)a - b'(t)]c, \quad \eta_2 = -a(k\sigma + \alpha) \tag{6}$$

iv) $\lambda = \lambda_0 \exp(\alpha\sigma)$

$$\xi_1 = ax + d, \quad \xi_2 = b(t), \quad \eta_1 = [a - b'(t)]c, \quad \eta_2 = -a/\alpha \tag{7}$$

Note that, in case ii, the equations become linear and the functions h(x,t) and g(x,t) satisfy the original equations, a feature of the linear systems. In the principal Lie algebra, there is one finite and one infinite parameter Lie Group of transformations. For the specific choice of filter functions, the finite and infinite parameter transformations increase.

3. SIMILARITY SOLUTIONS

Some similarity transformations and solutions will be constructed from the symmetries.

3.1. A general transformation

For the principal Lie Algebra corresponding to the arbitrary filter coefficient, the associated equations for similarity transformations can be written in their most general form as follows

$$\frac{dx}{a}=\frac{dt}{b(t)}=\frac{dc}{-b'(t)c}=\frac{d\sigma}{0} \tag{8}$$

The similarity variable and functions are defined as

$$\mu = x-\int\frac{a}{b(t)}dt,\qquad c=\frac{c(\mu)}{b(t)},\qquad \sigma=\sigma(\mu) \tag{9}$$

Under the above variables, the original equations are transformed to a nonlinear ODE system

$$c'+\lambda(\sigma)\,c=0,\qquad -a\sigma'+v\,c'=0 \tag{10}$$

The second equation can easily be integrated, but for the first equation to be integrated the specific form of the filter function should be known. In the following sections, more specific choice of transformations will be considered.

3.2. Translational Symmetry

All cases admit translational symmetries. If we choose ξ_1=a, ξ_2=b, η_1=0, η_2=0, then the similarity variable and functions read

$$\mu = x-mt,\qquad c=c(\mu),\qquad \sigma=\sigma(\mu) \tag{11}$$

where m=a/b, constant. This transformation reduces the original system to the following ODE system

$$c'+\lambda(\sigma)\,c=0,\qquad -m\sigma'+v\,c'=0 \tag{12}$$

For $\lambda=\lambda_o$, the final solutions are

$$c(x,t)=c_0\exp(-\lambda_0(x-mt)),\qquad \sigma(x,t)=\frac{c_0 v}{m}\exp(-\lambda_0(x-mt))+\sigma_0 \tag{13}$$

For $\lambda=\lambda_o\sigma$, the results are

$$c(x,t)=\frac{m\gamma_0/v}{1-\sigma_0\exp(-\lambda_0\gamma_0(x-mt))},\qquad \sigma(x,t)=\gamma_0\sigma_0\frac{\exp(-\lambda_0\gamma_0(x-mt))}{1-\sigma_0\exp(-\lambda_0\gamma_0(x-mt))} \tag{14}$$

3.3. Spiral Symmetry

One choice of spiral symmetry which is valid for all cases would be to select ξ_1=a, ξ_2=bt, η_1=-bc, η_2=0. For this specific choice, the similarity variable and functions are

$$\mu = t\exp(-mx),\qquad c=c(\mu)/t,\qquad \sigma=\sigma(\mu) \tag{15}$$

Under this transformation, the original equations read

$$-m\mu c'+\lambda(\sigma)\,c=0,\qquad \sigma'-m\,v\,c'=0 \tag{16}$$

For $\lambda=\lambda_o$, the solutions are

$$c(x,t)=\frac{\sigma_0}{mv}t^{(\lambda_0/m)-1}\exp(-\lambda_0 x),\qquad \sigma(x,t)=\sigma_0 t^{\lambda_0/m}\exp(-\lambda_0 x)+\gamma \tag{17}$$

and for a linear λ (i.e. $\lambda=\lambda_o\sigma$), the results are

$$c(x,t)=\frac{\gamma}{mv}\frac{c_0 t^{(\lambda_0\gamma/m)-1}\exp(-\lambda_0\gamma x)}{1-c_0 t^{\lambda_0\gamma/m}\exp(-\lambda_0\gamma x)},\qquad \sigma(x,t)=\frac{\gamma}{1-c_0 t^{\lambda_0\gamma/m}\exp(-\lambda_0\gamma x)} \tag{18}$$

A different choice of spiral transformation is allowed for the $\lambda=\lambda_o$ case, which is $\xi_1=a$, $\xi_2=bt$, $\eta_1=0$, $\eta_2=b\sigma$. The similarity variable and functions are

$$\mu = x - m\ln t, \qquad c = c(\mu), \qquad \sigma = t\sigma(\mu) \tag{19}$$

For this choice the final solutions are

$$c(x,t) = c_0 \exp(-\lambda_0 (x - m\ln t)),$$
$$\sigma(x,t) = \sigma_0 \exp(x/m) + \frac{c_0 v\lambda_0 t}{\lambda_0 m + 1}\exp(-\lambda_0 (x - m\ln t)) \tag{20}$$

3.4. Scaling Symmetry

The arbitrary function and constant filtration coefficient cases do not accept scaling symmetries. The remaining cases, however admit such a symmetry. Only a special sub-case corresponding to the linear filtration coefficient (i.e. $\lambda=\lambda_o\sigma$) will be considered. Choosing the infinitesimals as $\xi_1=ax$, $\xi_2=bt$, $\eta_1=-bc$, $\eta_2=-a\sigma$ yields

$$\mu = x/t^m, \qquad c = c(\mu)/t, \qquad \sigma = \sigma(\mu)/x \tag{21}$$

The final solutions are

$$c(x,t) = \frac{\gamma}{vt\left[1 - mc_0 (x/t^m)^{-\lambda_0\gamma/m}\right]}, \quad \sigma(x,t) = \frac{\gamma c_0 (x/t^m)^{-\lambda_0\gamma/m}}{x\left[1 - mc_0 (x/t^m)^{-\lambda_0\gamma/m}\right]} \tag{22}$$

Note that m is an arbitrary constant in the solutions and can be assigned specific values.

4. CONCLUDING REMARKS

Group-theoretic approach is applied to a nonlinear filtration equation used commonly in the literature. The filter coefficient is assumed to be an arbitrary function of the deposit. Group classification with respect to the filter function is done. The principal Lie Algebra is calculated. The specific forms of the functions for which symmetries increase has been determined and the symmetries corresponding to these cases are calculated. Similarity solutions are constructed for the special group transformations, namely the translational, spiral and scaling symmetries.

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POSSIBILISTIC DATA ENVELOPMENT ANALYSIS

İhsan Alp
Department of Statistics, Gazi University, Ankara, Turkey
ialp@fef.gazi.edu.tr

Abstract-Data of problems in real world, fuzziness/impreciseness/absence appears due to various reasons very often. In such cases, difficulty in model building can be overcome by using fuzzy set theory and concepts. In this study, only situation of data lacking will be considered. Data envelopment analysis cannot be used in the absence of one or more data in model. If those absent datum or data can be supplied with possibilistic membership function, the problem is solved. In this study, a suggestion will be given about how to apply data envelopment analysis with the assistance of possibilistic membership function in real world problem with lacking data. Model problem has been chosen in the banking sector, which is a very popular area.

Keywords- Data envelopment analysis (DEA), fuzzy mathematical programming, fuzzy decision making, possibilistic computation, α-cut, efficiency and banking.

1. INTRODUCTION

In present world, the sources being restricted and the ambitions being unlimited has made the thought of knowing and developing the efficiency of production units the most important matter to be solved. In the event concerned, in the case of multi input and output, things may get more complicated. Data envelopment analysis, which is developed to determine the efficiency of decision-making units, can overcome this difficulty with the aid of various models [1,2]. For that reason, it has been applied to many public and private sector problems successfully within a short period. All related input and output data should be in hand in order to apply the models of data envelopment analysis to a problem. In the case of data lacking in inputs and outputs, data envelopment analysis cannot be put into practice. However, it is likely to meet fuzziness/impreciseness/absence in the real world problems data [3]. Especially, in almost every situation involved human factor, data are not constant, crisp values. For example, a worker cannot do the same job at the same constant time always. He does it more or less within a tolerance of time intervals due to several factors. Likewise, in an inflationist environment prices change continual. The amount of matter in a package of product varies from box to box. All these examples emphasize fuzziness. Sometimes one or more data do not exist owing to various factors.

Data envelopment analysis can not be put into practice in the case of data lacking situation. This fuzziness/impreciseness/absence can be overcome by fuzzy set theory and concepts. In this study only data absence situation will be considered. In the case of data absence in model, these data should be obtained in a realistic way. One of the ways of overcoming this difficulty easily is to assign a constant and unique value, which is based on expert opinion. A more realistic way is to determine it with help of the most pessimistic, the most optimistic and the most possibilistic values, that is, triangular or trapezoidal or other possibilistic distribution of membership function.

Thus, for the absent data membership degrees in [0,1] interval and corresponding absent data values can be found in line with the step size. Values found for absent data can be used in model to get the efficiency scores of DMUs. Due to the variety of possible values of absent data, efficiency scores of some DMUs may also change. Furthermore, some efficient DMUs may be inefficient and reference sets of inefficient DMUs may probably change too.

2. EFFICIENCY MEASUREMENT TOOL: DATA ENVELOPMENT ANALYSIS

DEA is an approach which determines and compares the relative efficiency of the institutions of production that are homogenous organizations to some degree like bank offices, hospitals, universities, economies [4,5,6,7].

In the simplest case, the efficiency of a DMU having one input and one output can be explained as follows:

$$\text{Efficiency} = \text{output} / \text{input} \tag{1}$$

In the case of too many outputs and inputs the work may get complicated and the definition for efficiency becomes:

$$\text{Efficiency} = \text{weighted sum of outputs} / \text{weighted sum of inputs.} \tag{2}$$

In this last definition, unbiased and objective determining of weights have been a matter for a longtime. The method Charnes et al recommend is to get the weights by the following model [4]:

$$\begin{aligned} & \max\ h_o = \sum_{r=1}^{s} u_r y_{ro} / \sum_{i=1}^{t} v_i x_{io} \\ & \text{subject to} \\ & \qquad \sum_{r=1}^{s} u_r y_{rj} / \sum_{i=1}^{t} v_i x_{ij} \leq 1 \\ & \qquad \sum_{r=1}^{s} v_i / \sum_{i=1}^{t} v_i x_{io} \geq \varepsilon \\ & \qquad \sum_{r=1}^{s} u_r / \sum_{i=1}^{t} v_i x_{io} \geq \varepsilon \\ & \quad j=1,\ldots,n \quad r=1,\ldots,s \quad i=1,\ldots,t \end{aligned} \tag{3}$$

o : number of DMU which efficiency calculated,
y_{rj}: amount of output r from unit j,
x_{ij}: amount of input i to unit j,
v_i : the weight given to input i,
u_r : the weight given to output r,
s : number of output,
m : number of input,
n : number of decision making unit.

The ratio of input output h_o is the objective function by choosing optimal input weights. The first restriction guarantees, with the same weights, all DMUs' efficiency rates not to be bigger than unit size. Other restrictions are $\in>0$ any positive small number and makes model (3) defined on a closed set. If efficiency degree obtained at the result of solution is $h_o=1$, DMU is absolutely efficient.

Fractional model approach given in (3) is appropriate in terms of explanation but has difficulties in viewpoint of calculation. For that reason it can be changed into a suitable structure, a linear programming model. Objective function can be made linear while the value of denominator be chosen constant and numerator can be made maximum.

$$\max\ w_o = \sum_{r=1}^{s} m_r y_{ro}$$
$$\text{subject to} \qquad (4)$$
$$\sum_{i=1}^{t} v_i x_{io} = 1$$
$$\sum_{r=1}^{s} m_r y_{rj} - \sum_{i=1}^{t} v_i x_{ij} \leq 0$$
$$m_r \geq \varepsilon,\quad v_i \geq \varepsilon,\quad \varepsilon > 0$$
$$j=1,\dots,n \quad r=1,\dots,s \quad i=1,\dots,t$$

Thereby, traditional linear programming model is attained and calculating advantages are obtained. Model (4) is easy to solve by the software package in hand. As in the previous model, in model (4) o is the number of the DMU whose efficiency will be calculated. v_i and m_r shows the weights of input and output, which will maximize the efficiency, score of DMU, ε which is any positive small number makes the weights of input and output positive.

3. METHOD

As is discussed in the introduction, if an observation is not definitely determined, the DMU corresponding to it is not definite either and if the mentioned DMU is placed in the efficient frontier, the efficiencies of the other DMUs which take it as reference also vary in a parallel way to the variation in the efficient frontier. Therefore, when there is an absent datum, a hypothetical value assignment will not yield a positive result even if an expert opinion is referred. We suggest representing these values within fuzzy set theory with a possibilistic membership function.

If the values that the inputs (X_{ij}) and outputs (Y_{ij}) could take can be approximately estimated, it can be shown with $\tilde{X}_{ij}$ ve $\tilde{Y}_{ik}$ fuzzy sets having membership functions of $\mu_{\tilde{X}_{ij}}$ ve $\mu_{\tilde{Y}_{ik}}$ [8,9]

$$\tilde{X}_{ij} = \{x_{ij}, \mu_{\tilde{X}_{ij}}(X_{ij}) / x_{ij} \in S(\tilde{X}_{ij})\}$$
$$\tilde{Y}_{ij} = \{y_{ij}, \mu_{\tilde{y}_{ik}}(Y_{ij}) / y_{ik} \in S(\tilde{Y}_{ik})\} \tag{5}$$

here $S(\tilde{X}_{ij})$ and $S(\tilde{Y}_{ij})$ DMU j. output and k. input universal set indicates the supporting set of $\tilde{X}_{ij}$ ve $\tilde{Y}_{ij}$ and we can calculate the activity of DMU using the following formula:

$$\max \quad w_0 = \sum_r m_r \tilde{Y}_{r0}$$

subject to

$$\begin{aligned} & \sum_i v_i \tilde{X}_{i0} = 1 \\ \sum_r m_r \tilde{Y}_{rj} \; - \; & \sum_i v_i \tilde{X}_{ij} \leq 0 \\ & m_r \geq \varepsilon \\ & v_i \geq \varepsilon \\ j = 1,\dots,n \quad r = 1,\dots,s \quad & i = 1,\dots,t \end{aligned} \tag{6}$$

In the model above all the values are considered as fuzzy. When one datum is absent, depending on the other observations, the smallest, the biggest and median values are taken and triangular membership function can be constituted and those values meet the most pessimistic and the most optimistic cases as well as the most possible ones. Once we have constituted membership function for all data, we can arrive the conclusion using possibilistic DEA technique.

Basically, this technique depends on the principle of getting a set of possibilistic values set with α-cut approach and DEA technique.

Suppose:

$$\begin{aligned} (X_{ij})_\alpha &= \{x_{ij} \in S(\tilde{X}_{ij}) / \mu_{\tilde{X}_{ij}} \geq \alpha\} = [(X_{ij})_\alpha^L, (X_{ij})_\alpha^U] \\ &= \left[\min_{x_{ij}}\{x_{ij} \in S(\tilde{X}_{ij}) / \mu_{\tilde{X}_{ij}} \geq \alpha\}, \max_{x_{ij}}\{x_{ij} \in S(\tilde{X}_{ij}) / \mu_{\tilde{X}_{ij}} \geq \alpha\} \right] \end{aligned} \tag{7a}$$

$$\begin{aligned} (Y_{ik})_\alpha &= \{y_{ik} \in S(\tilde{Y}_{ik}) / \mu_{\tilde{Y}_{ik}} \geq \alpha\} = [(Y_{ik})_\alpha^L, (Y_{ik})_\alpha^U] \\ &= \left[\min_{y_{ik}}\{y_{ik} \in S(\tilde{Y}_{ik}) / \mu_{\tilde{Y}_{ik}} \geq \alpha\}, \max_{y_{ik}}\{y_{ik} \in S(\tilde{Y}_{ik}) / \mu_{\tilde{y}_{ik}} \geq \alpha\} \right] \end{aligned} \tag{7b}$$

Shown with (7a) and (7b) α-cut set of $\tilde{X}_{ij}$ ve $\tilde{Y}_{ij}$. These intervals show possibilities of α-cut corresponding inputs and outputs. Membership function of r. DMU's activity result can be defined by using Zadeh's expansion principle as follows:

$$\mu_{\tilde{w}_r}(z) = \sup_{x,y} \min\{\mu_{\tilde{X}_{ij}}(x_{ij}), \mu_{\tilde{Y}_{ik}}(y_{ik}). \forall i,j,k / z = w_r(x,y)\} \tag{8}$$

Here $w_r(x,y)$, to calculate α-cut level frontiers of $\tilde{w}_r(x,y)$ is found from model (4). For this aim following mathematical programming formulas with α parameter can be given.

$$(w_r)^L_\alpha = \min_{\substack{(X_{ij})^L_\alpha \le x_{ij} \le (X_{ij})^U_\alpha \\ (Y_{ik})^L_\alpha \le y_{ik} \le (Y_{ik})^U_\alpha \\ \forall i,j,k}} \left\{ \begin{array}{r} \max\ w_0 = \sum m_r y_{r0} \\ \text{subject to} \\ \sum_i v_i x_{i0} = 1 \\ \sum_r m_r y_{rj} - \sum_i v_i x_{ij} \le 0 \\ m_r \ge \varepsilon \\ v_i \ge \varepsilon \\ j = 1,\dots,n \quad r = 1,\dots,s \quad i = 1,\dots,t \end{array} \right\} \tag{9a}$$

$$(w_r)^U_\alpha = \max_{\substack{(X_{ij})^L_\alpha \le x_{ij} \le (X_{ij})^U_\alpha \\ (Y_{ik})^L_\alpha \le y_{ik} \le (Y_{ik})^U_\alpha \\ \forall i,j,k}} \left\{ \begin{array}{r} \max\ w_0 = \sum m_r y_{r0} \\ \text{subject to} \\ \sum_i v_i x_{i0} = 1 \\ \sum_r m_r y_{rj} - \sum_i v_i x_{ij} \le 0 \\ m_r \ge \varepsilon \\ v_i \ge \varepsilon \\ j = 1,\dots,n \quad r = 1,\dots,s \quad i = 1,\dots,t \end{array} \right\} \tag{9b}$$

Theoretically, when DMU's outputs in the objective function and the inputs of other DMUs are at the bottom frontier but the outputs of DMUs in the objective function and the outputs of other DMUs are at the top frontier, w_r is expected to take a small value. But when the case is contrary, it is expected to take a big value. As a result, two-step mathematical programming in (9) can be transformed to a one step DEA form as follows:

$$\begin{array}{rl} \max\ w_0 = \sum m_r (Y_{r0})^L_\alpha & \\ \text{subject to} & \\ \sum_i v_i (X_{i0})^U_\alpha = 1 & \\ \sum_r m_r (Y_{rj})^U_\alpha - \sum_i v_i (X_{ij})^L_\alpha \le 0, \ \textit{for DMU}_o & \\ \sum_r m_r (Y_{rj})^L_\alpha - \sum_i v_i (X_{ij})^U_\alpha \le 0, \ j = 1,\dots,n,\ i \ne 0 & \\ m_r \ge \varepsilon & \\ v_i \ge \varepsilon & \\ r = 1,\dots,s \quad i = 1,\dots,t & \end{array} \tag{10a}$$

$$
\begin{aligned}
&\max\ w_0 = \sum m_r (Y_{r0})_\alpha^U \\
&\text{subject to} \\
&\sum_i v_i (X_{i0})_\alpha^L = 1 \\
&\sum_r m_r (Y_{rj})_\alpha^L - \sum_i v_i (X_{ij})_\alpha^U \le 0,\ for\ DMU_o \\
&\sum_r m_r (Y_{rj})_\alpha^U - \sum_i v_i (X_{ij})_\alpha^L \le 0,\ j = 1,\dots,n,\ i \ne 0 \\
&m_r \ge \varepsilon \\
&v_i \ge \varepsilon \\
&r = 1,\dots,s \quad i = 1,\dots,t
\end{aligned}
\tag{10b}
$$

After this, $\mu_{\tilde{w}_r}$ membership function can be attained from $\left[(w_r)_\alpha^L,(w_r)_\alpha^U\right]$ for different α-cut levels. After bringing the method to this step, possibilistic DEA could be applied to banks example.

4. EXPERIMENTAL APPLICATION

As a result of global integration, it is not likely that a country or a region can remain isolated or unaffected from global events, likewise, global society structure cannot be thought to remain unaffected by the events occurring in one region or country. The best example of this is the deep impacts of the crisis, which burst out in the Far East and affected the entire world a few years ago, and preventive efforts of worldwide communities against the danger for an international crisis upon the depressing situation our country fell. When we approach the subject from an academic perspective, the financial system that is the center of crisis and the banks that are the main elements of this system, whether they function effectively is a crucial question. The answer of this question is important in terms of the formation of competition in the sector as well. Efficiency analysis is an important tool while answering this question but, when the data are absent, it must be completed by supportive techniques. The problems created by lacking data can be overcome by fuzzy approach and possibilistic DEA.

4.1. Data

Data constituting the base for our study are as in Table 1. Variables are consumer loan interest (CLI), vehicle purchase loan interest (VPLI) and residence loan interest (RLI) that the banks open to their customers. And similarly input variables are TL deposit account interest, USD deposit account interest and DM deposit account interest. When those data are examined, as is seen in Table 1., some decision making units data are absent in terms of used variables and they are shown as *,**. When we consider that efficiency analysis is done over all observed data, this absent data create a fuzziness that affects the results of all DMUs. A bank manager, who wants to see his banks situation in the sector, cannot reach his aim unless he puts the values of those lacking data in their places. Lack of those data, may be because of the banks' lacking data in terms of mentioned variables, Halk Bankası (VPLI and RLI), MNG Bank (RLI), Vakıfbank (RLI), Ziraat Bankası (RLI) not being able to fulfill the service for their customers or due to some other reasons.

Table 1. Data

DMU NO	BANKS	OUTPUTS (%)			INPUTS (%)		
		CLI	VPLI	RLI	TL	USD	DM
1	Akbank	5,25	4,95	5,25	25,89	12,00	12,00
2	Anadolubank	8,50	7,50	7,50	20,04	10,00	10,00
3	Demirbank	6,50	5,75	5,75	20,88	9,50	7,00
4	Denizbank	6,50	5,50	7,00	29,33	6,75	4,50
5	Dışbank	9,50	7,50	7,50	39,25	8,00	7,50
6	Emlakbank	6,00	5,50	6,00	26,00	9,00	7,00
7	Esbank	6,25	6,00	6,00	34,24	7,50	6,50
8	Finansbank	6,50	5,00	5,00	28,39	10,00	7,50
9	Garanti Bankası	5,75	5,25	5,25	35,91	9,00	8,00
10	Halk Bankası	12,00	*,**	*,**	25,05	9,00	8,00
11	İş Bankası	5,75	5,25	5,25	34,24	7,00	6,00
12	Kentbank	7,75	7,25	8,00	26,72	8,50	7,50
13	Koçbank	6,95	5,65	5,65	36,74	825	6,75
14	MNG Bank	9,50	8,50	*,**	33,40	9,50	8,00
15	Osmanlı Bankası	8,50	5,95	5,95	33,40	7,50	5,25
16	Oyakbank	5,95	5,45	5,45	37,58	11,00	9,25
17	Pamukbank	6,50	5,75	5,75	29,23	10,00	9,00
18	Sitebank	9,50	8,00	8,00	29,23	12,00	12,00
19	Şekerbank	5,90	5,75	6,00	30,06	8,50	8,50
20	TEB	7,50	6,50	6,75	32,57	8,50	7,50
21	Toprakbank	5,50	5,00	5,00	32,57	9,00	9,00
22	Türk Ticaret Bankası	5,75	5,25	5,25	28,39	7,50	6,50
23	Vakıfbank	6,00	6,00	*,**	28,00	7,25	6,50
24	Yapı Kredi Bankası	5,25	5,00	5,25	26,72	9,50	7,50
25	Ziraat Bankası	5,95	5,95	*,**	25,05	7,25	5,75
Minimum		5,25	4,95	5,00	20,04	6,75	4,50
Maximum		12,00	8,50	8,00	39,25	12,00	12,00
Average		7,00	6,00	6,00	29,95	8,88	7,72

Even the data are fuzzy for some banks; it is out of expectation that those banks declare lower vehicle or residence loan interest than the lowest level of other banks or higher than the highest level. Therefore, we can equate the worst value to the observed minimum, the best value to the observed maximum and the optimum value to the observed average interest rate. These defined three values and Halk Bank's vehicle loan interest variable $\tilde{Y}_{\text{Halk BankasıaVPLI}} = [4.95, 6{,}00, 8{,}50]$ and again Halk Bank's MNG Bank's and Vakıfbank's residence loan variable can be shown by $\tilde{Y}_{\text{Halk Bank,RLI}} = \tilde{Y}_{\text{MNG Bank,RLI}} = \tilde{Y}_{\text{Vakıakıfba RLI}} = [5.00, 6.00, 8.00]$ fuzzy numbers. Besides, their possibilistic membership functions

$$\mu_{\tilde{Y}_{\text{Halk BankasıaVPLI}}}(y)=\begin{cases}(y-4.95)/(6.00-4.95) & 4.95\le y\le 6.00\\ (8.50-y)/(8.50-6.00) & 6.00\le y\le 8.50\end{cases} \quad (11)$$

$$\begin{aligned}\mu_{\tilde{Y}_{\text{Halk Bankası, RLI}}}(y)&=\mu_{\tilde{Y}_{\text{MNG Bank, RLI}}}(y)=\mu_{\tilde{Y}_{\text{Vakıakıfba, RLI}}}(y)\\ &=\begin{cases}(y-5.00)/(6.00-5.00) & 5.00\le y\le 6.00\\ (8.00-y)/(8.00-6.00) & 6.00\le y\le 8.00\end{cases}\end{aligned} \quad (12)$$

and also α-cuts of membership function, which we defined, are as follows:

$$\begin{aligned}&\left[(Y_{\text{Halk Bankas ı VPLI}})^{L}_{\alpha},(Y_{\text{Halk Bankas ıaVPLI}})^{U}_{\alpha}\right]=[4.95+1.05\alpha,8.50-2.5\alpha]\\ &\left[(Y_{\text{Halk Bankas ı, RLI}})^{L}_{\alpha},(Y_{\text{Halk Bankas ı, RLI}})^{U}_{\alpha}\right]=\left[(Y_{\text{MNG Bank, RLI}})^{L}_{\alpha},(Y_{\text{MNG Bank, RLI}})^{U}_{\alpha}\right]\\ &\qquad=\left[(Y_{\text{Vak ıakıfba RLI}})^{L}_{\alpha},(Y_{\text{Vak ıakıfba RLI}})^{U}_{\alpha}\right]\\ &\qquad=[5+\alpha,8-2\alpha].\end{aligned} \quad (13)$$

After obtaining the membership functions of all absent data using model (10) efficiency scores can be calculated.

4.2. Solution and Results

Following the model's setting up as in the above model (10), for 0, 0.25, 0.50, 0.75, 1.00 levels of α-cut, from expression (13) according to lower and upper limit values (Table 2.) for each DMUs the model was rewritten and solutions were done and the results are shown in Table 3. Although all technology coefficients are assumed as fuzzy, as is said above other known values excluding data lacking units that are expressed by (*) can be expressed by possibilistic membership function lower and upper limits are congruent for all α-cut values. Upper and lower limit values for various α-cut levels of variables VPLI and RLI containing impreciseness/absence are as shown in Table 2.

Table 2.Upper and lower limit values for different α-cut levels

	VPLI		RLI	
α	Upper	lower	Upper	lower
0.00	4,95	8,50	5,00	8,00
0.25	5,21	7,88	5,25	7,50
0.50	5,48	7,25	5,50	7,00
0.75	5,74	6,63	5,75	6,50
1.00	6,00	6,00	6,00	6,00

Obtained results of the solutions for different α levels are given to the service of the decision maker in the form of a table (Table 3.). What is outstanding in the table is that fuzzy values are rather efficient on efficiency scores of both DMUs they originate from and others. The value that Ziraat Bankası, 25. numbered DMU and Vakıfbank 23. numbered DMU will take in RLI variable value which is absent has directly affected not only theirs but also others' efficiency. When we look at the table, in the situation of taking values with Ziraat Bankası's α's 0.00, 0.25, 0.50, Vakıfbank's α's 0.00 and 0.25 level about upper limit, that is 8,00, 7.50 and 7.00 they take place in the efficient

frontier and also for α=0.00 the mentioned banks entering efficient frontier, Kentbank which is 12 numbered DMU at the efficient frontier will lose its efficiency and take place in the group of inefficient DMUs with a DEA score of 0.957. Another point seen in the table is that for α=1 level which is the most possible situation, the mentioned banks are not efficient but fit the upper and lower limits in this level.
When approached to this level, a decrease at the upper limits of efficiency scores of these banks and an increase at the lower limits are observed. It is vice versa with the other banks.

Table 3. Possibilistic DEA results

DMU	α=0,00		α=0,25		α=0,50		α=0,75		α=1,00	
	lower	upper	lower	upper	lower	upper	lower	upper	Lower	upper
1	0,564	0,556	0,564	0,561	0,564	0,564	0,564	0,564	0,564	0,564
2	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
3	0,936	0,838	0,935	0,882	0,934	0,925	0,933	0,929	0,932	0,932
4	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
5	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
6	0,805	0,740	0,804	0,777	0,804	0,800	0,803	0,802	0,803	0,803
7	0,885	0,867	0,884	0,883	0,884	0,883	0,884	0,883	0,883	0,883
8	0,723	0,610	0,720	0,646	0,715	0,675	0,711	0,692	0,706	0,706
9	0,654	0,629	0,654	0,648	0,654	0,651	0,653	0,652	0,653	0,653
10	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
11	0,832	0,814	0,832	0,832	0,832	0,832	0,832	0,832	0,832	0,832
12	1,000	0,957	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
13	0,778	0,758	0,778	0,776	0,778	0,778	0,778	0,778	0,778	0,778
14	1,000	0,976	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
15	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
16	0,578	0,541	0,577	0,567	0,577	0,573	0,576	0,575	0,576	0,576
17	0,701	0,622	0,700	0,661	0,699	0,690	0,698	0,694	0,697	0,697
18	0,867	0,799	0,865	0,862	0,864	0,846	0,862	0,854	0,860	0,860
19	0,774	0,740	0,774	0,769	0,774	0,773	0,774	0,774	0,774	0,774
20	0,869	0,835	0,869	0,861	0,869	0,866	0,869	0,867	0,868	0,868
21	0,632	0,598	0,631	0,622	0,631	0,629	0,631	0,630	0,631	0,631
22	0,797	0,758	0,797	0,787	0,797	0,792	0,796	0,794	0,795	0,795
23	0,921	1,000	0,922	1,000	0,924	0,981	0,926	0,944	0,929	0,929
24	0,689	0,627	0,689	0,663	0,688	0,688	0,689	0,689	0,688	0,688
25	0,977	1,000	0,978	1,000	0,981	1,000	0,983	0,993	0,986	0,986

5. COMMENTS

Banks constituting an important leg of the financial system is a significant base of the economy of a country. Productive working of these institutes is of vital significance for it as well. Especially, if the environment is fuzzy, it is unavoidable for the decision maker to make decisions. From this perspective, fuzzy DEA appears as a good problem solving alternative and guide that the decision maker can refer to. In this study, solution has been done only for 5 (0.00, 0.25, 0,50, 0.75, 1.00) level of α. In a real decision making environment, decision maker will definitely ask for more detailed information to give more absolute decisions and want a to change in 0.10 or perhaps 0.05 step sizes. In this case, decision alternatives will naturally increase and the decision maker will get opportunities for better decisions.

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