DIRECT INTEGRALS OF HILBERT SPACES AND VON NEUMANN ALGEBRAS

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Contents

1. Operator Valued Measurable Functions

The theory of direct integrals of von Neumann algebras will rely heavily on the notion of an operator valued measurable function. So we will need to establish some results on Banach and Operator Valued Measurable functions first. Also, many facts about direct integrals require the use of a measurable cross-section theorem, which can only be apply to *Borel* functions. Because of this, we will need to worry about the case of measurable functions as well as Borel measurable functions.

1.1. **Measurable Functions Into Banach Spaces.** Before we talk about the special case of Operator or Hilbert valued measurable functions, it will be helpful to handle some general facts about Banach valued measurable functions. In particular, we will need this material, when we discuss measurable functions into the Banach space of trace class operators. In this section we will need to prove a collection of technical results, analogous to the classical case before we discuss the theory of Operator Valued Measurable functions as well as direct integrals. Fortunately, most results one expects to work out do, but one has to repeat most of the work that goes into standard measure theory. All this work will pay off, but the results may seem technical at first and we will have to do a lot of grunt work before getting to the really interesting material Throughout the entire discussion our measure spaces

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to have a locally compact Hausdorff space topology. We are only really concerned with standard Borel spaces, so this is not a big assumption.

Definition 1.1.1. Let X be a separable Banach space and μ a Radon measure on a locally compact Hausdorff space Y. A function $f\colon Y \to X$ is μ-measurable if the following two conditions hold:

(i)for all $\phi \in X^*, y \to \phi(f(y))$ is μ-measurable

(ii) for every compact $K \subseteq Y$, there is μ-measurable set $E \subseteq K$ such that $f(E)$ is separable and $\mu(K \setminus E) = 0$.

Note that since a Banach space X has a topology, it makes sense to talk about Borel sets in X (the sets in the σ-algebra generated by open sets), we will can also talk about weakly Borel sets in X (the sets in the σ-algebra generated by weakly open sets). We can also talk about Borel functions into X. We will need some basic propositions about measurable and Borel Banach valued functions.

Proposition 1.1.2. *Let X be a Banach space and μ a Radon measure on a locally compact Hausdorff space Y. Then*

(i) The sum of two μ-measurable X-valued functions is μ-measurable, also if $\lambda \in \mathbb{C}$ and $f\colon Y \to X$ is μ-measurable, then so if λf. If X is separable, then the sum of two Borel functions is Borel and λf is Borel for all Borel $f\colon Y \to X$ and $\lambda \in \mathbb{C}$.

Let $f_n\colon Y \to X$ be a sequence of μ-measurable functions. (ii) If $f_n(x) \to g(x)$ pointwise almost everywhere, then g is μ-measurable.

Assume X is separable.

(iii) If $f\colon Y \to X$ is μ-measurable, then so is $\|f\|$. If f is Borel, then so is $\|f\|$.

(iv) If $f_n\colon Y \to X$ are Borel, then the set E of points where f_n converges is a Borel set and defining $g(x) = \lim_{n\to\infty} f_n(x)$ when $x \in E$ and 0 otherwise, is a Borel function.

(v) The Borel sets coincide with the weakly Borel sets.

(vi) A function $f\colon Y \to X$ is Borel if and only if for every $\phi \in X^*$ we have that $\phi \circ f$ is Borel.

Proof. For $A \subseteq X$ we will use $N_\varepsilon(A) = \{x \in X : \|x - y\| < \varepsilon, \text{ for some } y \in A\}$ it is an open set.

(i) Let $f, g\colon Y \to X$ be μ-measurable. It is clear that $\phi \circ (f + g)$ is μ-measurable for all $\phi \in X^*$. Let $K \subseteq Y$ be compact. Choose N_1, N_2 null such that $f(K \setminus N_1), g(K \setminus N_2)$ are separable, set $N = N_1 \cup N_2$. Then

$$f + g(K \setminus N) \subseteq f(K \setminus N_1) + g(K \setminus N_2)$$

since the sum of two separable sets is separable we are done. A similar proof works for $\lambda f, \lambda \in \mathbb{C}$.

Now suppose f, g as above are Borel, and X is separable. Let $F \subseteq X$ be closed, and let (x_n) be a countable dense sequence in X. Then $f(y) + g(y) \in F$ if and only if for every $\varepsilon > 0$ there exists n, k such that $\|f(y) - x_n\|, \|g(y) - x_k\| \leq \varepsilon$ and $x + y \in N_\varepsilon(F)$. Thus

$$(f + g)^{-1}(F) = \bigcap_{n=1}^{\infty} \bigcup_{k,l=1, x_k + x_l \in N_\varepsilon(F)}^{\infty} f^{-1}(B(x_n, \varepsilon)) \cap g^{-1}(B(x_k, \varepsilon))$$

and this is a Borel set.

(ii) The fact that $\phi \circ g$ is μ-measurable for all $\phi \in X^*$ follows from basic measure theory. Suppose $K \subseteq Y$ is compact, and for each n choose N_n null such that $f_n(K \setminus N_n)$ is measurable. Let N' be the set of points where f_n does not converge to g, then

$$N = N' \cup \bigcup_{n=1}^{\infty} N_n$$

is null. We claim that $g(K \setminus N)$ is separable. Since the closure of a countable union of separable spaces is separable, it suffices to show that

$$g(K \setminus N) \subseteq \overline{\bigcup_{n=1}^{\infty} f_n(K \setminus N_n)}.$$

So let $y \in K \setminus N$ and $\varepsilon > 0$. Then since $y \notin N'$ we can find k such that

$$\|g(y) - f_k(y)\| < \varepsilon.$$

Since $y \notin N_k$ we have that $f_k(y) \in f_k(K \setminus N_k) \subseteq \bigcup_{n=1}^{\infty} f_n(K \setminus N_n)$, since $\varepsilon > 0$ is arbitrary, this verifies the claim.

(iii) First note that there is a countable sequence (ϕ_n) in X^* such that $\|\phi_n\| = 1$ and the weak* closed convex hull of the ϕ_n is $\{\psi \in X^* : \|\psi\| \leq 1\}$. Indeed, let (x_n) be a dense sequence in X and, by the Hahn-Banach Theorem, choose $\phi_n \in X^*$ such that $\|\phi_n\| = 1$ and $\phi_n(x_n) = \|x_n\|$, let C be the weak*-closed convex hull of the ϕ_n. If $\phi \in X^*$, $\|\phi\| = 1$ and $\phi \notin C$, then, by the Hahn-Banach Theorem, there exists real numbers $\alpha < \beta$ and $\psi \colon X^* \to \mathbb{C}$ weak*-continuous such that

$$\mathrm{Re}(\psi(\phi_n)) < \alpha < \beta < \mathrm{Re}(\psi(\phi))$$

for all n. Since ψ is weak* continuous there exists $x \in X$ such that $\psi(\phi) = \phi(x)$. Thus

$$\mathrm{Re}(\phi_n(x)) < \alpha < \beta < \mathrm{Re}(\phi(x))$$

for all n. Choose $x_{n_k} \to x$, then

$$|\|x_{n_k}\| - \phi_{n_k}(x)| = |\phi_{n_k}(x - x_{n_k})| \leq \|x - x_{n_k}\| \to 0.$$

Since $\|x_{n_k}\| \to \|x\|$ we have

$$\|x\| \leq \alpha < \beta < |\phi(x)|$$

and this contradicts the fact that $\|\phi\| = 1$.

So let (ϕ_n) be as above. Then the Hahn-Banach Theorem implies that

$$\|f(x)\| = \sup_n |\phi_n(f(x))|$$

and this is μ-measurable, and if f is Borel the above implies that $\|f(x)\|$ is Borel.

(iv) Since X is complete, the set of points where f_n converges is the same as the set of points where it is Cauchy. Thus

$$E = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k,l \geq m} \{x : \|f_k(x) - f_l(x)\| < 1/n\}$$

this is a Borel set by (i) if each f_n is Borel. Further if $F \subseteq X$ is closed and $x \in E$, then $g(x) \in F$ if and only if for all $\varepsilon > 0$ $f_k(x)$ is at distance at most ε from F for all large k. Thus

$$g^{-1}(F) = E\left(\cap \bigcap_{n=1}^{\infty} \bigcup_{m=1} \bigcap_{k=m}^{\infty} f_k^{-1}(N_\varepsilon(F))\right) \cup E^c \text{ if } 0 \in F$$

and

$$g^{-1}(F) = E \cap \bigcap_{n=1}^{\infty} \bigcup_{m=1} \bigcap_{k=m}^{\infty} f_k^{-1}(N_\varepsilon(F)) \text{ if } 0 \notin F.$$

These are Borel sets, so g is Borel.

(v) Since weakly open sets are open it is clear that the weakly Borel sets are Borel.

Let ϕ_n be the sequence in X^* constructed as in (iii). For $x, y \in X, \varepsilon > 0$ note that the Hahn-Banach Theorem implies that $\|x-y\| \leq \varepsilon$ if and only if $|\phi(x)-\phi(y)| \leq \varepsilon$ for all $\phi \in X^*$ with $\|\phi\| = 1$. Since the closed convex hull of the ϕ_n are weak* dense in the unit ball of X^* we have that $\|x - y\| \leq \varepsilon$ if and only if $|\phi_n(x - y)| \leq \varepsilon$ for all n. Thus

$$\overline{B(x,\varepsilon)} = \bigcap_{n=1}^{\infty} \{y : |\phi_n(x-y)| \leq \varepsilon\}$$

so $\overline{B(x,\varepsilon)}$ is weak*-dense. The seperability (or more appropriately second countability) of X shows that every open set is a countable union of sets of the form $\overline{B(x,\varepsilon)}$ and this proves (v).

(vi) If $f : Y \to X$ is Borel, then for all $\phi \in X^*$ and $U \subseteq \mathbb{C}$ open we have

$$(\phi \circ f)^{-1}(U) = f^{-1}(\phi^{-1}(U))$$

and since $\phi^{-1}(U)$ is open ,we have that $f^{-1}(\phi^{-1}(U))$ is Borel. Conversely, suppose $\phi \circ f$ is Borel for all $\phi \in X^*$. As above, let (ϕ_n) be a sequence in X^* such that $\|\phi_n\| = 1$ and the weak*-closed convex hull of the ϕ_n is the norm closed unit ball of X^*. Then as in (v) we have

$$\overline{B(x,\varepsilon)} = \bigcap_{n=1}^{\infty} \{y : |\phi_n(x-y)| \leq \varepsilon\} = \bigcap_{n=1}^{\infty} \phi_n^{-1}(\overline{B(\phi_n(x),\varepsilon)})$$

thus

$$f^{-1}(\overline{B(x,\varepsilon)}) = \bigcap_{n=1}^{\infty} f^{-1}(\phi_n^{-1}(\overline{B(\phi_n(x),\varepsilon)}))$$

as in (v), this shows that f is Borel.

□

Corollary 1.1.3. *Let X be a Banach space and μ a Radon measure on a locally compact Hausdorff space Y, let $f : Y \to X$ be μ-measurable and $\phi : Y \to \mathbb{C}$ μ-measurable. Then ϕf is μ-measurable. If X is separable and f as above is Borel, and ϕ as above is Borel, then ϕf is Borel.*

Proof. Let us handle the μ-measurable case first. Approximating ϕ by simple functions, we may assume that ϕ is a simple function. By the proceeding proposition, it suffices to show that $\chi_E f$ is μ-measurable for all $E \subseteq Y$ measurable. Fix E and f, and let $K \subseteq Y$ be compact, choose $F \subseteq K$ such that $\mu(K \setminus F) = 0$ and $f(F)$ is separable. Then

$$(\chi_E f)(F) \subseteq f(F) \cup \{0\},$$

hence is separable. The equation

$$\phi(\chi_E(x)f(x)) = \chi_E(x)\phi(f(x))$$

shows that $\phi \circ \chi_E f$ is measurable whenever $\phi \circ f$ is measurable and $\phi \in X^*$. Thus $\chi_E f$ is measurable.

Assume X is seperable, as above all we have to show is that $\chi_E f$ is Borel for all Borel $E \subseteq Y$. But by the above proposition we know that f is Borel if and only if $\phi \circ f$ is Borel for all $\phi \in X^*$ (since this is the same as requiring that f is weakly Borel). As above the equation

$$\phi \circ (\chi_E f) = \chi_E \phi \circ f$$

shows that $\chi_E f$ is Borel.

□

Proposition 1.1.4. *Let X be a Banach space and μ a Radon measure on a σ-compact locally compact Hausdorff space Y. Let $f\colon Y \to X$. Then f is μ-measurable if and only if for any compact $K \subseteq Y$ and $\varepsilon > 0$, there is a compact $C \subseteq K$ such that $f\big|_C$ is continuous and $\mu(K \setminus C) < \varepsilon$.*

If X is separable, then f is μ-measurable if and only if there exists a Borel map $g\colon Y \to X$, such that $g = f$ almost everywhere.

Proof. Let $Y = \bigcup_{n=1}^{\infty} K_n$ with K_n compact.

Suppose the condition about compact sets and ε's holds. If $\phi \in X^*$ then for each n, by assumption, we can find $C_n \subseteq K_n$ compact, such that

$$(a)\mu(K_n \setminus C_n) < \frac{1}{2^n}$$

$$(b) f\big|_{C_n} \text{ is continuous.}$$

By Tietze Extension Theorem, we may find $h_n\colon Y \to \mathbb{C}$ continuous such that $h_n = \phi \circ f$ on C_n. Let

$$N = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{x \in Y : h_k(x) \neq \phi \circ f(x)\}.$$

Note that if $x \notin N$ there is an n such that $h_k(x) = f(x)$ for all $k \geq n$. In particular we may define $g(x)$ to be the limit of the $h_n(x)$ when it exists, and 0 otherwise, and this will be a Borel function. Further we will have $g(x) = \phi \circ f(x)$ off N, so to show $\phi \circ f$ is measurable, it suffices to show that N has measure 0. To do this, it suffices to show that $\mu(N \cap K_n) = 0$, for every n. But

$$\mu(N \cap K_n) = \lim_{m \to \infty} \mu \left(K_n \cap \bigcup_{k=m}^{\infty} \{x \in Y : h_k(x) \neq f(x)\} \right)$$

and for all large k,

$$\mu \left(K_n \cap \bigcup_{k=m}^{\infty} \{x \in Y : h_k(x) \neq f(x)\} \right) \leq \sum_{k=n}^{\infty} 1/2^k \to 0$$

as $m \to \infty$.

Let $K \subseteq Y$ be compact, choose $C_n \subseteq K$ compact such that $\mu(K \setminus C_n) \le 1/2^n$, and f is continuous on C_n Set $E = \bigcup_{n=1}^{\infty} C_n$. Then $\mu(K \setminus E) = 0$ and

$$f(E) = \bigcup_{n=1}^{\infty} f(C_n).$$

Each $f(C_n)$ is compact, since f is continuous on C_n. Thus $f(C_n)$ is separable, being compact metric, so $f(E)$ is separable, being a countable union of separable spaces.

Conversely, suppose that $f\colon Y \to X$ is μ-measurable and $\varepsilon > 0, K \subseteq Y$ is compact. Choose $E \subseteq K$ such that $\mu(K \setminus E) = 0$ and $f(E)$ is separable. Let X_0 be the closed linear span of $f(E)$ and let (x_n) be a dense sequence in X_0. The same proof as in the above proposition shows that $g_n(x) = \|f(x) - x_n\|$ is measurable on E, and hence on K since $\mu(K \setminus E) = 0$. By measure theory, we can find $C_n \subseteq K$ compact such that $\mu(K_n \setminus C) < \varepsilon/2^n$ and $g_n\big|_{C_n}$ is continuous. Set $C = \bigcap_{n=1}^{\infty} C_n$, then C is compact, and $\mu(K \setminus C) \le \varepsilon$, and on C each $g_n(x)$ is continuous. If $y \in X$, then $\|f(x) - y\|$ is a uniform limit of the g_n (choose $x_{n_k} \to y$), and is thus continuous. Fix $y_0 \in C$, by the preceding $y \to \|f(y) - f(y_0)\|$ is continuous on C, and $y \to f\big|_C(y)$ is continuous at y_0. Thus $f\big|_C$ is continuous.

Now suppose X is separable. If f is almost everywhere equal to a Borel function, it is clear that $\phi \circ f$ is μ-measurable for all $\phi \in X^*$, so f is μ-measurable. Conversely, if f is μ-measurable hen for each n, by assumption, we can find $C_n \subseteq K_n$ compact, such that

$$(a)\mu(K_n \setminus C_n) < \frac{1}{2^n}$$

$$(b) f\big|_{C_n} \text{ is continuous.}$$

By Tietze Extension Theorem, we may find $h_n\colon Y \to X$ continuous such that $h_n = \phi \circ f$ on C_n. If we define $g(x)$ to be the limit of $h_n(x)$ when it exists and 0 otherwise, then the proceeding proposition shows that g is Borel. The same argument as above show that $g = f$ almost everywhere, so we are done.

□

We wish to extend analogous facts about simple functions from the the current situation.

Definition 1.1.5. Let X be a Banach space and μ a Radon measure on a σ-compact locally compact Hausdorff space Y. A function $f\colon Y \to X$ is a simple function if there exist disjoint measurable sets $Y_1, \dots, Y_n$ in Y and $x_1, \dots, x_n \in X$ such that $f(y) = \sum_{j=1}^{n} \chi_{Y_j}(x) x_j$.

Proposition 1.1.6. *Let X be a Banach space and μ a Radon measure on a σ-compact locally compact Hausdorff space Y. Let $f\colon Y \to X$, if f is μ-measurable function then there exists simple functions $(\phi_n)_{n\in\mathbb{N}}$ such that $f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \phi_n(x)$ for almost every x. If f is Borel and X is separable, then we can choose Borel simple functions $(\phi_{mn})_{m,n\in\mathbb{N}}$ such that $f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \phi_{mn}(x)$ for every x. Further, if $\|f(x)\| \le R$ for some $R > 0$, then we may force the ϕ_{mn} (either Borel or μ-measurable depending upon the case) so that $\|\phi_{mn}(x)\| \le R$ as well.*

Proof. Let $Y = \bigcup_{n=1}^{\infty} K_n$ with the K_n compact and $K_n \subseteq K_{n+1}$.

We first handle the case when f is continuous, and show that we can choose a sequence of simple functions which converge to f uniformly on compact sets. By compactness, for each n we can find $x_1^{(n)}, \dots, x_{k_n}^{(n)}$ in $f(K_n)$ which are $1/2^n$ dense. Let $A_j^{(n)} = \{y \in Y : \|\phi(y) - x_j^{(n)}\| < 1/2^n\}$, and define $B_j^{(n)}$ by

$$B_1^{(n)} = A_1^{(n)}, B_j^{(n)} = A_j^{(n)} \setminus \left(\bigcup_{l=1}^{j-1} A_l^n \right).$$

Set

$$\phi_n(x) = \sum_{j=1}^{n} \chi_{B_j^{(n)}} x_j^{(n)}.$$

For $x \in K_n$ we have that $f(x) \in B_j^{(n)}$ by construction and $\|\phi_n(x) - x\| \le 1/2^n$. Thus $\phi_n(x) \to f(x)$ uniformly on K_n. If $\|f(x)\| \le R$, then defining $y_j^{(n)}$ by $R\frac{x_j^{(n)}}{\|x_j^{(n)}\|}$ if $\|x_n\| > R$ and setting

$$\psi_n(x) = \sum_{j=1}^{n} \chi_{B_j^{(n)}} y_j^{(n)}$$

we see that $\|\psi_n(x)\| \le R$ and still $\psi_n(x) \to \phi_n(x)$ uniformly on compact sets.

Now suppose $f : Y \to X$ is μ-measurable. Then we can choose $C_n \subseteq K_n$ compact such that $\mu(K_n \setminus C_n) < 1/2^n$ and $f\big|_{C_n}$ is continuous. By the first step we can choose a simple function ϕ_n which is zero outside C_n and bounded by R if f is such that

$$\|\phi_n(x) - f(x)\| \le 1/2^n$$

for $x \in C_n$. Set $N = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} X \setminus C_n$, we see exactly as in 1.1.4 that N has measure zero, and that $\phi_n(x) \to f(x)$ for $x \notin N$.

Now suppose f is Borel and that X is separable. Let (x_n) be a countable dense subset of X. Let

$$A_j^{(n)} = f^{-1}(B(x_j, 1/2^n)), B_1^{(n)} = A_1^{(n)}, B_j^{(n)} = A_j^{(n)} \setminus \left(\bigcup_{l=1}^{j-1} A_l^{(n)} \right).$$

Then $B_j^{(n)}$ are Borel sets, and if we set

$$\phi_{mn}(x) = \sum_{j=1}^{m} \chi_{B_j^{(n)}}(x) x_j$$

it is easy to see that $B_j^{(n)}$ have the desired properties. If $\|f(x)\| \le R$ for every x, then we replace x_n with a countable dense subset of $B(0, R)$ and run the same construction. □

For later use, we will also need the generalizations of L^p spaces to Banach valued measurable functions.

Definition 1.1.7. Let X be a separable Banach space and μ a Radon measure on a locally compact Hausdorff space Y. For $1 \le p < \infty$, let $L^p(Y, X, \mu)$ be the set of all $f : Y \to X$ μ-measurable such that

$$\int_Y \|f\|^p \, d\mu < \infty$$

We only proved measurability of $\|f\|$, for f measurable, when X is seperable. This is why we need this assumption. This is how separability will be used in the proofs below as well. We shall use

$$\|f\|_p = \left(\int_Y \|f\|^p \, d\mu\right)^{1/p}$$

for the norm of f in $L^p(Y, X, \mu)$.

Theorem 1.1.8. *Let X be a separable Banach space and μ a Radon measure on a locally compact Hausdorff space Y, and $1 \leq p < \infty$. Then $L^p(Y, X, \mu)$ is a Banach space under the norm $\|f\|_p$. The set of continuous functions from Y to X with compact support is dense in $L^p(Y, X, \mu)$ as well as the simple functions from Y to X. Further, if $f_n \to f$ in L^p, then there exists a subsequence f_{n_k} of f such that $f_{n_k} \to f$ almost everywhere.*

Proof. From the standard Minkowski Inequality it follows that $L^p(Y, X, \mu)$ is a normed space. Suppose $f_n \in L^p(Y, X, \mu)$ is such that

$$\sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Then

$$\int_Y \left(\sum_{n=1}^{\infty} \|f_n\|\right)^p d\mu = \int_Y \lim_{N\to\infty} \left(\sum_{n=1}^{N} \|f_n\|\right)^p d\mu \leq$$
$$\liminf \int_Y \left(\sum_{n=1}^{N} \|f_n\|\right)^p d\mu.$$

By Minkowski's inequality we know that

$$\liminf \int_Y \left(\sum_{n=1}^{N} \|f_n\|\right)^p d\mu \leq \left(\sum_{n=1}^{N} \|f_n\|_p\right)^p \leq \left(\sum_{n=1}^{\infty} \|f_n\|_p\right)^p < \infty.$$

These inequalities tell us that

$$\sum_{n=1}^{\infty} \|f_n(x)\| < \infty$$

almost everywhere. Since X is complete we have an almost everywhere defined function $f(x) = \sum_{n=1}^{\infty} f_n(x)$. By the same inequalities as above

$$\int_Y \|f - \sum_{n=1}^{N} f_n\|^p \, d\mu = \int_Y \|\sum_{n=N}^{\infty} f_n\|^p \, d\mu \leq \left(\sum_{n=N}^{\infty} \|f_n\|_p\right)^p \to 0$$

as $N \to \infty$. This proves completeness of $L^p(Y, X, \mu)$.

We show that the continuous functions with compact support from Y to X are dense. By the dominated convergence theorem

$$\int \|f - f\chi_{\{\|f\| \leq N\}}\|^p \, d\mu = \int_{\{\|f\| > N\}} \|f\|^p \, d\mu \to 0$$

as $N \to \infty$. So we may assume that f is bounded. Again by the dominated convergence theorem we know that

$$\int \|f - f\chi_{\{\|f\| > 1/n\}}\|^p \, d\mu = \int \|f\chi_{\{\|f\| \leq 1/n\}}\|^p \, d\mu \to 0$$

and by Chebyshev $\{\|f\| > 1/n\}$ has finite measure. Thus we may assume that f is bounded and supported on a set of finite measure. Since Radon measures are inner regular on all sets of finite measure (see [1] Proposition 7.5), we can find compact sets $K_n \subseteq \{x : f(x) \neq 0\}$ such that $\mu(\{x : f(x) \neq 0\} \setminus K_n) \to 0$. As above this implies that $\|f - \chi_{K_n} f\| \to 0$, and this we may assume that f is bounded and compactly supported, let K be the support of f. Let $R > 0$ be such that $\|f\| \leq R$, and given $\varepsilon > 0$, choose $C \subseteq K$ compact, such that $f\big|_C$ is continuous, and $\mu(K \setminus C) < \varepsilon$. By Tietze Extension Theorem, we may find a compactly supported $g \colon K \to B_X(0, R)$ which is continuous and such that $g\big|_Y = f$. Then

$$\int_Y \|f - g\|^p \, d\mu = \int_C \|f - g\|^p \, d\mu \leq 2R^p \varepsilon,$$

since R is fixed and $\varepsilon > 0$ is abritrary this shows that the continuous functions with compact support are dense in $L^p(Y, X, \mu)$. The case of simple functions is similar, as above we may assume that Y is compact and that f is bounded by $R > 0$. Then we use the proceeding proposition to find $\phi_n \colon Y \to R$ simple, such that $\|f - \phi_n\|_p \to 0$.

Finally suppose that $\|f_n - f\|_p \to 0$, by Chebyshev's inequality for every $\varepsilon > 0$

$$\mu(\{\|f - f_n\| \geq \varepsilon\}) \to 0.$$

Thus we may choose an increasing sequence of integers $\{n_k\}$ such that

$$\mu(\{\|f - f_m\| \geq 1/2^k\}) \leq 1/2^k$$

if $m \geq n_k$. Let

$$N = \bigcap_{k=1}^{\infty} \bigcup_{m=n_k}^{\infty} \{\|f - f_m\| \geq 1/2^k\}$$

as in Proposition 1.1.4 we have that N is a null set and $f_{n_k} \to f$ on $Y \setminus N$, this completes the proof. □

1.2. **Measurable Operator Valued Functions.** In this section, we will primarily be concerned with measurable functions $f \colon Y \to B(H)$, with H a Hilbert space. We could use the Banach space structure of $B(H)$ to define measurability of such a function. But we saw in the preceding section, that we typically have good theorems about measurability of functions when the Banach space we are mapping into is separable. Since $B(H)$ is not separable in the cases we are primarily concerned with, we shall use a different notion of measurability. Before introducing it, we shall note one fact.

Proposition 1.2.1. *Let μ be a Radon measure on a locally compact Hausdorff space Y. Let H be a Hilbert space and $T \colon Y \to B(H)$ be such that $x \to T(x)\xi$ is μ-measurable for all $\xi \in H$. Then for all $\xi \colon Y \to H$ μ-measurable we have $x \to T(x)\xi(x)$ is μ-measurable. Similarly if H is separable, and $T \colon Y \to B(H)$ is such that $x \to T(x)\xi$ is Borel for all $\xi \in H$, then for all Borel $\xi \colon Y \to H$ we have that $x \to T(x)\xi$ is Borel.*

Proof. By proposition 1.1.6 we may approximate ξ, pointwise almost everywhere by a sequence $\xi_n \colon Y \to H$ of simple μ-measurable functions. Since μ-measurability is closed under pointwise almost everywhere limits, we may assume that ξ is a simple function. In this case, our assumption and Proposition 1.1.2 and Corollary 1.1.3 imply that $T(x)\xi$ is μ-measurable. The proof of the Borel case is the same. □

Definition 1.2.2. Let μ be a Radon measure on a locally compact Hausdorff space Y and H a Hilbert space. We say that $T\colon Y \to B(H)$ is strongly* μ-measurable, if for all $\xi \in H$ both $x \to T(x)\xi$ and $x \to T(x)^*\xi$ are μ-measurable. Similarly we say that $T\colon Y \to B(H)$ is strongly* Borel if for all $\xi \in H$, both $x \to T(x)\xi$ and $x \to T(x)^*\xi$ are Borel.

By the above proposition, strongly* μ-measurable maps are closed under pointwise multiplication as well as pointwise adjoints. The same remark applies to strongly* Borel maps, provided that the Hilbert space in question is separable.

The next proposition lists some basic facts about operator valued measurable maps. As in the Banach case, by a simple function $\phi\colon Y \to B(H)$ we mean a function of the form $\sum_{j=1}^n \chi_{A_j} T_j$ with A_j disjoint measurable sets and $T_j \in B(H)$, similarly we define Borel simple functions.

Proposition 1.2.3. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y. Let H be a separable Hilbert space and $T\colon Y \to B(H)$.*

(i) If T is strongly* μ-measurable, then $\|T\|$ is measurable. If T is strongly* Borel, then $\|T\|$ is Borel.

(ii) If T is strongly* μ-measurable, then there is a sequence $T_n\colon Y \to B(H)$ of simple functions such that $T_n \to T$ in the strong* topology. That is, $T_n\xi \to T\xi$ and $T_n^*\xi \to T^*\xi$ for all $\xi \in H$. If T is Borel, then there are $T_{mn}\colon Y \to B(H)$ Borel simple functions such that $\lim_n \lim_m T_{mn} = T$ in the strong* topology. If $\|T\| \leq R$ for some $R > 0$, then in either case we may assume that $\|T_n\| \leq R$.

(iii) If $T_n\colon Y \to B(H)$ are strongly* Borel, then $E = \{x : T_n(x)$ converges in the strong* topology$\}$ is a Borel set, and if we set $S(x) = \lim_{n\to\infty} T_n(x)$, for $x \in E$ and zero for $x \notin E$, then S is strongly* Borel.

(iv) If $T_n\colon Y \to B(H)$ are strongly* μ-measurable and $T_n \to S$ pointwise almost everywhere then S is strongly * μ-measurable.

Proof. (i) First assume that T is strongly* μ-measurable. Let (ξ_n) be a dense sequence in $\{\xi \in H : \|\xi\| = 1\}$. By assumption $x \to T(x)\xi_n$ is measurable for all n, and thus Proposition 1.1.2 implies that $\|T(x)\xi_n\|$ is measurable. Thus

$$\|T\| = \sup_n \|T(x)\xi_n\|$$

is measurable. The same proof works for the Borel case.

(ii) Let $Y = \bigcup_{n=1}^\infty K_n$ with K_n compact and $K_n \subseteq K_{n+1}$. Let P_n be a sequence of finite rank projections which converge strongly to 1. Set $T_n = P_n T P_n$, then T_n is strongly* μ-measurable. Since $P_n H P_n$ is finite-dimensional, we may regard T_n as a map from a fintie dimensional Hilbert space to itself. Regarding T_n as a matrix, we see that we can find $\phi_{m,n}(x)$, simple functions with values in $B(PnHP_n)$ such that $\phi_{m,n}(x) \to P_n T P_n$ pointwise (by applying the standard result to each matrix entry). By Egoroff's Theorem and inner regularity, we can find $C_n \subseteq K_n$ compact,$m_n \in N$ such that $\mu(K_n \setminus C_n) < 1/2^n$ and $\|\phi_{m_n,n}(x) - T(x)\| < 1/2^n, \|\phi_{mn}^*(x) - T(x)\| < 1/2^n$ on C_n (here we use the uniqueness of a vector space topology on a finite dimensional space). As before, if

$$N = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty C_n^c$$

then N has measure zero, and $\psi_n = \phi_{m_n,n}$ converges in the strong* topology to T_n. If $\|T_n\| \leq R$, write $\psi_n = \sum \chi_{A_j^{(n)}} S_j$ with $A_j^{(n)}$ disjoint for fixed j, redefine ψ_n by replacing S_j with $R\frac{S_j}{\|S_j\|}$ if $\|S_j\| > R$. Since ψ_n, ψ_n^* are uniformly close to T_n, T_n^* on C_n we see that ψ_n still converges pointwise almost everywhere to T_n in the strong* topology.

For the Borel case, let P_n be as above. As above we can find simple Borel functions $\phi_{mn}\colon Y \to B(p_n H P_n)$ which converge pointwise to $P_n T P_n$ as $m \to \infty$. Since $P_n \to 1$, this shows that $\lim_n \lim_m \phi_{mn} = T$ pointwise in the strong*-topology. If $\|T\| \leq R$, then if we choose a basis for $P_n H$ and write $P_n T P_n$ as a matrix, we see that all of its entries have absolute valued at most R. Thus, standard measure theory tells us the we may force ϕ_{mn} to converge uniformly in norm to $P_n T P_n$ on K_n. Again, writing

$$\phi_{mn} = \sum \chi A_j^{(mn)} S_j$$

with $A_j^{(mn)}$ disjoint for fixed j, and replacing S_j with $R\frac{S_j}{\|S_j\|}$ if $\|S_j\| > R$, shows that we may force $\|\phi_{mn}\| \leq R$.[1] □

For the next proposition we will investigate how strongly* behave with respect to measurable maps into trace class-operators. Recall that the set of trace-class operators are those $S \in B(H)$ such that

$$\mathrm{Tr}(|S)) = \sum_i \langle |S| e_i, e_i \rangle < \infty$$

with e_i an orthonormal basis of H. We shall use $B(H)_*$ for the set of trace-class operators. Recall $B(H)_*$ is a vector space wit norm $\|S\|_1 = \mathrm{Tr}(|S|)$, and that $B(H)_*$ is complete in this norm. Further, $B(H)_*$ is separable if H is separable. Finally, recall that $B(H) = (B(H)_*)^*$ under the duality

$$\langle T, S \rangle = \mathrm{Tr}(TS)$$

with $T \in B(H), S \in B(H)_*$.

(iii) We first establish that E is a Borel set. Let (ξ_n) be a dense sequence in H. We claim that

$$E = \left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x : \|T_k(x)\| \leq N\} \right) \bigcap_{l=1}^{\infty} \{x : T_n(x)\xi_l, T_n(x)^*\xi_l \text{ converge } \}.$$

Indeed if x is in the set on the right hand side, then $\|T_n(x)\|$ is bounded and $T_n(x)\xi_k$ converges for all k. Since ξ_k are dense, standard arguments show that T_n converges in the strong* topology. Conversely, if $T_n(x)$ converges in the strong* topology, then we have that $\|T_n(x)\|$ is bounded, by the principal of uniform boundedness. Also $T_n(x)\xi_l, T_n^*(x)\xi_l$ converges for all l, and so x is in the set on the right hand side. This shows that E is Borel by Proposition 1.1.2. Once we know that E is Borel the fact that T is strongly* Borel follows from the definition and Proposition 1.1.2.

(iv) This is obvious from the definition and Proposition 1.1.2.

Corollary 1.2.4. *Let Y be a σ-compact locally compact Hausdorff space and μ a Radon measure on Y, and let H be a separable Hilbert space. For any $T\colon Y \to B(H)$, strongly* μ-measurable, there is a $T'\colon Y \to B(H)$ strongly *-Borel such that $T = T'$ almost everywhere.*

---

[1] Again this uses that ϕ_{mn} is uniformly close to $P_n T P_n$ on K_n

Proof. By the above proposition, we can find $T_n \colon Y \to B(H)$ simple functions such that $T_n \to T$ in the strong* topology almost everywhere. Write

$$T_n = \sum_j \chi_{A_j^{(n)}} T_{nj}$$

with $A_j^{(n)}$ measurable and $T_{nj} \in B(H)$. Let $B_j^{(n)} \subseteq A_j^{(n)}$ be Borel and such that $\mu(B_j^{(n)} \setminus A_j^{(n)}) = 0$, (this is possible since Y is σ-compact). Let $T_n' = \sum_j \chi_{A_j^{(n)}} T_{nj}$, and let $T'(x) = \lim_{n\to\infty} T_n'(x)$, when the limit exists and zero otherwise. By the above proposition T' does is strongly* Borel, and we see that $T' = T$ almost everywhere.

□

Proposition 1.2.5. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y. Let H be a separable Hilbert space and $T \colon Y \to B(H), S \colon Y \to B(H)_*$.*

(i) If S is μ-measurable[2], then S^* is μ-measuarble. Similarly, if S is Borel, then S^* is Borel.

(ii) If S is μ-measurable, then when regard as a map into $B(H)$ it is strongly μ-measurable. Similarly, if S is Borel, then regarded as a map into $B(H)$, it is strongly* Borel.*

(iii) If S is μ-measurable, and T is strongly μ-measurable, then ST and TS are μ-measurable as a map into $B(H)_*$. Similarly, if S is Borel, and T is strongly* Borel, then ST and TS is Borel as a map into $B(H)_*$.*

Proof. (i) Since $B(H)_*$ is separable, by the definition of μ-measurability and Proposition 1.1.2 (vi) all we have to do is check that that $x \to \mathrm{Tr}(AS(x)^*)$ is μ-measurable/Borel whenever S is and $A \in B(H)$. But

$$\mathrm{Tr}(AS(x)^*) = \overline{\mathrm{Tr}(A^* S(x))}$$

so this is clear.

(ii) By (i), all we have to show is that $\xi \to S(x)\xi$ is μ-measurable/Borel whenver S is. But with

$$\xi \otimes \eta^*(\zeta) = \langle \zeta, \xi \rangle \eta$$

we have that

$$\langle S(x)\xi, \eta \rangle = \mathrm{Tr}((\xi \otimes \eta^*) S(x))$$

and thus $\langle S(x)\xi, \eta\rangle$ is μ-measurable/Borel for all $\xi, \eta \in H$. Since H is separable, this implies that $S(x)\xi$ is μ-measurable/Borel whenever S is.

(iii) Since $(TS)^* = S^* T^*$, parts (i) and (ii) tell us we only have to check whether TS is μ-measurable/Borel. Assume first that S is μ-measurable and T is strongly* μ-measurable. By Proposition 1.1.6 we can find simple functions $S_n \colon Y \to B(H)_*$ such that $\|S_n(x) - S(x)\|_1 \to 0$ for almost every x. If we can show each TS_n is μ-measurable we will be done. Since μ-measurablility is closed under sums it thus suffices to show that TA when $A \in B(H)_*$ is fixed, since we already know that $\chi_E S$ is μ-measurable for all $E \subseteq Y$ measurable by Corollary 1.1.3. Let $A \in B(H)_*$, then we can find $(\lambda_n) \in l^1(\mathbb{N})$ and orthonormal vectors ξ_n, η_n such that

$$A\zeta = \sum_{n=1}^{\infty} \lambda_n \langle \zeta, \xi_n \rangle \eta_n.$$

[2] As a map into the Banach space $B(H)_*$.

Thus

$$T(x)A = \sum_{n=1}^{\infty} \lambda_n (T(x)\eta_n \otimes \xi_n^*$$

with the sum converging in the $\|\cdot\|_1$ norm. So it suffices to show that $T(x)\eta \otimes \xi^*$ is μ-measurable for all $\eta, \xi \in H$. But, for all $B \in B(H)$ we have

$$\mathrm{Tr}((T(x)\eta \otimes \xi^*)B) = \langle BT(x)\eta, \xi\rangle = \langle T(x)\eta, B^*\xi\rangle,$$

and since T is strongly* μ-measurable, this is a measurable function. Thus $T(x)\eta \otimes \xi^*$ is μ-measurable. The proof for the Borel case is the same, we simply express $S = \lim_m \lim_n S_{mn}$ with S_{mn} simple and Borel, and repeat the argument.

□

For later use we note the following proposition, which shows that given a Borel map $T\colon Y \to B(H)$ we have Borel maps $Y \times B \to B(H), Y \times B(H)_* \to \mathbb{C}$, given by multiplication and evaluation, when $B \subseteq B(H)$ is bounded.

Proposition 1.2.6. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y. Let H be a separable Hilbert space and $T\colon Y \to B(H)$ a strongly* Borel map.*

(i) The strong* topology on $\overline{B(0,R)} \subseteq B(H)$ is separable and completely metrizable (i.e. there is a complete metric on $\overline{B(0,R)}$ inducing the strong*-topology).

(ii) The map $Y \times \overline{B(0,R)} \to B(H)$ given by $(x, S) \to T(x)S$ is Borel when $\overline{B(0,R)}$ is given the strong* topology.

(iii) The map $Y \times B(H)_* \to \mathbb{C}$ given by $(x, A) \to Tr(T(x)A)$ is Borel.

Proof. Fix a dense sequence (ξ_n) in the norm closed unit ball of H.

(i) Define

$$d(S,T) = \sum_{n=1}^{\infty} \frac{1}{2^n}\|(S-T)\xi_n\| + \sum_{n=1}^{\infty} \frac{1}{2^n}\|(S^*-T^*)\xi_n\|,$$

then d is a metric on $\overline{B(0,R)}$. We claim that d is complete and gives the strong*-topology. Suppose S_k in $\overline{B(0,R)}$ is Cauchy with respect to d. Then $S_k\xi_n, S_k^*\xi_n$ are Cauchy for each n. If $\xi \in H$ and $\|\xi\| = 1$ and $\varepsilon > 0$ is given, we can find n such that $\|\xi_n - \xi\| < \varepsilon$ and K such that $k, l \geq K$ imply

$$\|S_k\xi_n - S_l\xi_n\| < \varepsilon, \|S_k^*\xi_n - S_l^*\xi_n\| < \varepsilon.$$

Then for $k, l \geq K$ we have

$$\|S_k\xi - S_l\xi\| \leq 2R\varepsilon + \|S_k\xi_n - S_l\xi_n\| < (2R + 1_\varepsilon$$

and thus $S_k\xi$ is Cauchy for $\|\xi\| = 1$. Similarly $S_k^*\xi$ is Cauchy for $\|\xi\| = 1$, by scaling $S_k\xi, S_k^*\xi$ are Cauchy for each $\xi \in H$. Thus we can define

$$S\xi = \lim_{k\to\infty} S_k\xi$$

$$T\xi = \lim_{k\to\infty} S_k^*\xi.$$

We claim that $T = S^*$. Indeed, for any $\xi, \eta \in H$ we have

$$\langle S\xi, \eta\rangle = \lim_{k\to\infty}\langle S_k\xi, \eta\rangle = \lim_{k\to\infty}\langle \xi, S_k^*\eta\rangle = \langle \xi, T\eta\rangle$$

thus $T = S^*$. Standard arguments show that $d(S_k, S) \to 0$, so d is complete.

The argument above also shows that if S_i is a net such that $d(S_i, S) \to 0$ and $\|S_i\| \leq R$, then $S_i \to S$ in the strong* topology. Conversely suppose $\|S_i\| \leq R$

and $S_i \to S$ in the strong* topology, in particular $\|S\| \leq R$. If $\varepsilon > 0$ is given, then choose n such that $2^{-n} < \varepsilon$. Then we have

$$d(S_i, S) \leq 8R\varepsilon + \sum_{n=1}^{N} \frac{1}{2^n}\|S_i\xi_n - S\xi_n\| + \sum_{n=1}^{N} \frac{1}{2^n}\|S_i^*\xi_n - S^*\xi_n\|,$$

since $S_i \to S$ in the strong* topology, we have that

$$\limsup_i d(S_i, S) \leq 8R\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof of (i).

(ii) By Proposition 1.2.3 we may assume that T is a simple function, and since strong* Borel maps form a $*$-algebra, we may assume that $T(x) = \chi_A(x)B$ for some $A \subseteq Y$ Borel and $B \in B(H)$ with $\|B\| \leq R$. In this case the map in question is

$$(x, S) \to \chi_A(x)BS.$$

Since $(x, S) \to S$ is strongly*-Borel and $(x, S) \to \chi_A(x)$ is Borel it follows that $(x, S) \to \chi_A(x)BS$ is strongly$*$-Borel, and this completes the proof.

(iii) As in (ii), we may assume that $T(x) = \chi_E(x)B$ for some $E \subseteq Y$ Borel and $B \in B(H)$. In this case we are considering

$$(x, S) \to \mathrm{Tr}(\chi_E(x)BS) = \chi_E(x)\,\mathrm{Tr}(BS)$$

since $S \to BS$ is continuous and $x \to \chi_E(x)$ is Borel, it follows that $(x, S) \to \mathrm{Tr}(\chi_E(x)BS)$ is Borel, as desired. □

2. Direct Integrals of Hilbert Spaces and Decomposable Operators

2.1. **Direct Integrals of Hilbert Spaces.** We already know the notion of a direct sum of Hilbert spaces, in many cases it is useful to use a more continuous version of this. For this reason we discuss the notion of a measurable field of Hilbert spaces and direct integrals of Hilbert spaces.

Definition 2.1.1. Let μ be a Radon measure on the σ-compact locally compact Hausdorff space Y. A *measurable field* of Hilbert spaces is a collection $\{H_x : x \in Y\}$ of Hilbert spaces together with a linear subspace $\mathcal{S}$ of $\prod_{x\in X} H_x$, whose elements are called *measurable sections* satisfying the following axioms:

(i) If $\eta \in \prod_{x\in X} H_x$, then $\eta \in \mathcal{S}$ if and only if $x \to \langle \xi(x), \eta(x)\rangle$ is measurable for all $\xi \in \mathcal{S}$. (ii) There is a sequence $(\xi_n(x))$ in $\mathcal{S}$ such that for almost every $x \in X$, the closed linear span of $\xi_n(x)$ is H_x.

Definition 2.1.2. Let μ be a Radon measure on the σ-compact locally compact Hausdorff space Y, and H_x a measurable field of Hilbert spaces over Y. We define *the direct integral of* H_x,denoted $\int_Y^{\oplus} H_x\, d\mu(x)$ to be the set of all $\xi \in \mathcal{S}$ (modulo agreeing on measure zero sets) such that $\int_Y \|\xi(x)\|^2\, d\mu(x) < \infty$. On $\int_Y^{\oplus} H_x\, d\mu(x)$ we have the inner product

$$\langle \xi, \eta\rangle = \int_Y \langle \xi(x), \eta(x)\rangle_{H_x}\, d\mu(x).$$

By the same argument as in Theorem 1.1.8, we see that $\int_Y^{\oplus} H_x\, d\mu(x)$ is a Hilbert space. Before proceeding to examples we need one Lemma.

Lemma 2.1.3. *Let μ be a Radon measure on the σ-compact locally compact Hausdorff space Y. Let $\{H_x : x \in X\}$ be Hilbert spaces and suppose there exists $\xi_n \in \prod_{x\in X} H_x$ such that*

(i) $x \to \langle \xi_n(x), \xi_m(x)\rangle$ is measurable, for all $x \in X$.

(ii) For almost every $x \in X$, the space H_x is closed linear span of $\{\xi_n(x) : n \in \mathbb{N}\}$.

Then $x \to \dim H_x$ is measurable and there exists $\eta_n \in \prod_{x\in X} H_x, n \in \mathbb{N}$ with the following properties:

(a) $x \to \langle \xi_n(x), \eta_m(x)\rangle$ is measurable for all $m, n \in \mathbb{N}$

(b) If $\eta \in \prod_{x\in X} H_x$ has $x \to \langle \xi_n(x), \eta(x)\rangle$ measurable for all $n \in \mathbb{N}$, then $x \to \langle \eta_n(x), \eta(x)\rangle$ is measurable for all m.

(c) $\langle \eta_n(x), \eta_m(x)\rangle = \delta_{mn}$ for $m, n \le \dim H_x$

(d) $\eta_n(x) = 0$ for $n > \dim H_x$

(e) H_x is the closed linear span of $\{\eta_n(x) : n \in \mathbb{N}\}$.

Proof. We construct η_n inductively. Suppose that $\eta_1, \dots, \eta_k$ have been constructed so that $(a), (b)(c), (d)$ are satisfied for $\eta_1, \dots, \eta_k$. Let

$$P_k(x)\xi = \sum_{j=1}^{k} \langle \xi, \eta_j(x)\rangle \eta_j(x)$$

then ${}_kP(x)$ is the projection onto the closed linear span of $\eta_1(x), \dots, \eta_k(x)$. Let

$$A_j = \{x : (1 - P_k(x))\xi_j \neq 0\}$$

$$B_m = A_m \setminus \left(\bigcup_{k=1}^{m-1} A_k \right).$$

Set

$$\eta_{k+1}(x) = \sum_{j=k+1}^{\infty} \frac{\chi_{B_j}(x)}{\|(1 - P(x))\xi_j(x)\|} (1 - P(x))\xi_j(x).$$

Then, $\eta_1, \dots, \eta_{k+1}$ satisfy $(a), (b), (c), (d)$. Finally the sequence $(\eta_n(x))$ satsifies (e), since we are simply appying the Gramm-Schmidt process pointwise. Further we have that

$$\dim H_x = \sum_{k=1}^{\infty} \|\eta_k(x)\|^2$$

which proves that $\dim H_x$ is measurable. □

Corollary 2.1.4. *Let μ be a Radon measure on the σ-compact locally compact Hausdorff space Y. Let $\{H_x : x \in X\}$ be Hilbert spaces. Suppose that $(\xi_n)_{n\in\mathbb{N}}$ in a sequence in $\prod_{x\in X} H_x$ such that $x \to \langle \xi_n(x), \xi_m(x)\rangle$ is measurable for all $n, m \in \mathbb{N}$, and such that H_x is the closed linear span of $\{\xi_n(x) : n \in \mathbb{N}\}$ for almost every x. Let $\mathcal{S}$ be the set of all $\eta \in \prod_{x\in X} H_x$ such that $\langle \xi_n(x), \eta(x)\rangle$ is measurable for all $n \in \mathbb{N}$. Then with $\mathcal{S}$ as measurable sections we have that H_x is a measurable field of Hilbert spaces over Y.*

Proof. We only have to show that $\xi \in \mathcal{S}$ if and only if $x \to \langle \xi(x), \eta(x)\rangle$ is measurable for all $\eta \in \mathcal{S}$. Let (η_n) be a sequence in $\prod_{x\in X} H_x$ satisfying $(a) - (e)$ in the proceeding proposition. Suppose $\xi \in \prod_{x\in X} H_x$ has $x \to \langle \xi(x), \eta(x)\rangle$ for all $\eta \in \mathcal{S}$. Then

$$\langle \xi(x), \xi_k(x)\rangle = \sum_{n=1}^{\infty} \langle \xi(x), \eta_n(x)\rangle \langle \eta_n(x), \xi_k(x)\rangle$$

by $(a),(b)$ of the proceeding proposition we have that $\xi \in \mathcal{S}$. Conversely, suppose $\xi, \eta \in \mathcal{S}$. Then

$$\langle \xi(x), \eta(x) \rangle = \sum_{n=1}^{\infty} \langle \xi(x), \eta_n(x) \rangle \langle \eta_n(x), \eta(x) \rangle,$$

so $(a),(b)$ of the proceeding proposition imply that $x \to \langle \xi(x), \eta(x) \rangle$ is measurable.

□

Example 2.1.5. Let X be a Riemann manifold, then T_xX is a measurable field of Hilbert spaces over X. Indeed, using second countability of X, we can cover X by open set U_n for which there are smooth vector fields $\xi_1^{(n)}, \dots, \xi_{\dim X}^{(n)} \in \prod_{x \in U_n} T_xX$ which pointwise give an orthonormal basis for T_xX. Setting $B_n = A_n \setminus \left(\bigcup_{j=1}^{n-1} U_j \right)$ and setting $\eta_j(x) = \sum_{n=1}^{\infty} \chi_{B_n}(x) \xi_j^{(n)}$ gives measurable sections which satisfy the proceeding proposition. In this case, one can show that $\int_X^{\oplus} T_xX \, d\mu(x)$ consists of all Borel vector fields $\xi \in \prod_{x \in X} T_xX$ (Borel as a map into TX) such that

$$\int_Y \|\xi(x)\|^2 \, d\mu(x) < \infty$$

Where μ is the measure we get from the Riemann metric. We can run the same construction replacing T_xX with k-forms at x, or even $L^2(T_xX)$ with respect to measure on T_xX we get from the Riemann metric, similarly we can perform fiber-wise operations (direct sum,tensor product) provided these give us an inner product at every point.

Example 2.1.6. Let A be a separable C^*-algebra, and let Y be a σ-compact locally compact Hausdorff space. Suppose that $\phi \colon Y \to A^*$ is such that $x \to \phi(x)(a)$ is measurable for all $a \in A$ and $\|\phi\| \leq 1$. Let $L^2(A, \phi(x))$ be the GNS space for $\phi(x)$. If a_n is a dense sequence in A, then using $\xi_n(x) = a_n \in L^2(A, \phi(x))$ we see that $L^2(A, \phi(x))$ is a measurable field of Hilbert spaces. A similar remark holds when A is replaced by a von Neumann algebra M and A^* with M_*, provided M_* is separable.

Example 2.1.7. Let G be a second-countable locally compact group, and and let Y be a σ-compact locally compact Hausdorff space with Radon measure μ. Suppose $(\pi_y)_{y \in Y}$ are cyclic unitary representations of G on H_y (e.g. π_y could be irreducible) with cyclic vectors ξ_y, such that $\langle \pi_y(x)(g)\xi_y, \xi_y \rangle$ is measurable for all $g \in G$. Using vector fields $\eta_n(y) = \pi_y(x_n)\xi_m$ with x_n a dense sequence in G, gives H_y the structure of a measurable field of Hilbert spaces.

Example 2.1.8. Let Y be a σ-compact locally compact Hausdorff space with Radon measure μ, and let H be a fixed separable Hilbert space. The constant field is given by setting $H_x = H$ for all $x \in Y$. The measurable sections are precisely all $\xi \colon Y \to H$ such that $\langle \xi(y), \eta \rangle$ is measurable for all $\eta \in H$. In this case $\int_Y^{\oplus} H \, d\mu(y)$ is just $L^2(Y, H, \mu)$.

This last example is actually the most important, as the following theorem shows.

Theorem 2.1.9. *Let Y be a σ-compact locally compact Hausdorff space with Radon measure μ, and let H_y be a measurable field of Hilbert spaces over Y. Then there exists disjoint sets $(Y_n)_{n \in \mathbb{N} \cup \{0,\infty\}}$ in Y such that $Y_\infty \cup \bigcup_{n=0}^{\infty} Y_n$ is conull, and* *unitary operators* $U_n(y)\colon \mathbb{C}^n \to H_y$, *with* $y \in Y_n$ *(using* $\mathbb{C}^\infty = l^2(\mathbb{N})$ *such that* $y \to \chi_{Y_n}(y)U_n^*(y)\xi(y)$ *is measurable for all* $\xi \in \mathcal{S}$. *Thus* U_n *induces a unitary isomorphism between* $\int_{Y_n}^{\oplus} H_y\, d\mu(y)$ *and* $L^2(Y_n, \mu, \mathbb{C}^n)$.

Proof. Let $\eta_n(x)$ be as in Lemma 2.1.3, by Lemma 2.1.3 we know that $n(x) = \dim H_x$ is measurable. Thus we can set $Y_n = \{y \in Y : \dim H_y = n\}$, for $y \in Y_n$, if we define

$$U_n(y)\colon \mathbb{C}^n \to H_y$$

by

$$U_n(y)\left(\sum_{j=1}^n c_j e_j\right) = \sum_{j=1}^n c_j \eta_j(y)$$

then $U_n(y)$ has the desired properties.

□

Because of the proceeding theorem, when discussing direct integrals we can often reduce to the case of $L^2(Y, H, \mu)$. Finally, we note one important case of how Direct Integrals appear naturally.

Theorem 2.1.10. *Let* Y *be a* σ*-compact locally compact Hausdorff space with Radon measure* μ, *and let* $\pi\colon L^\infty(Y,\mu) \to H$ *be a nondegenerate normal∗ representation on a separable Hilbert space* H. *Then there is a measurable field* H_y *of Hilbert spaces such that* π *is unitarily equivalent to the representation* ρ *of* $L^\infty(Y,\mu)$ *on* $\int_Y^{\oplus} H_y\, d\mu(y)$ *given by*

$$\rho(f)\xi(y) = f(y)\xi(y).$$

Proof. By Zorn's Lemma and the separability of H there exists a countable set J and unit vectors $(\xi_j)_{j\in J}$ such that $\overline{L^\infty(Y,\mu)\xi_j} \perp \overline{L^\infty(Y,\mu)\xi_k}$ if $j \neq k$ and

$$\bigoplus \overline{L^\infty(Y,\mu)\xi_j} = H.$$

Since π is normal, we can find $g_j \in L^1(Y,\mu)$ such that

$$\langle \pi(f)\xi_j, \xi_j\rangle = \int_Y f g_j\, d\mu.$$

Let $E_j = \{x : g_j(x) \neq 0\} <$ and let

$$n(x) = \sum_{j=1}^{\infty} \chi_{E_j}(x).$$

Let π_j be the restriction of π to $\overline{L^\infty(Y,\mu)\xi_j)}$. Considering the unitary $Uf = fg_j^{1/2}$ we see that π_j is unitary equivalent to the representation of $L^\infty(Y,\mu)$ on $L^2(E_j,\mu)$ given by multiplying by $\chi_{E_j} f$. Thus π is isomorphic to the representation of $L^\infty(Y,\mu)$ on

$$\bigoplus_{j\in J} L^2(E_j,\mu).$$

So we will work with this representation instead. Set $H_y = \mathbb{C}^{n(y)}$ with $\mathbb{C}^\infty = l^2(\mathbb{N})$, set $Y_n = \{x : n(x) = n\}$. We show that π is isomorphic to the representation of $L^\infty(Y,\mu)$ on

$$\oplus_{n=1}^{\infty} L^2(Y_n, \mathbb{C}^{n(y)}, \mu)$$

given by multiply $\xi(y)$ by $f(y)\xi(y)$. As the above Hilbert space is

$$\int_Y \mathbb{C}^{n(y)}\,d\mu(y)$$

and L^∞ acts appropriately on this space this will complete the proof. Since the representation of $L^\infty(Y,\mu)$ is unitarily equivalent to

$$\bigoplus_{j\in J} L^2(E_j,\mu) = \bigoplus_{j\in J} L^2(E_j,\mu) \bigoplus_{n\in\mathbb{N}} L^2(E_j\cap Y_n,\mu) \cong \bigoplus_{n=1}^{\infty}\bigoplus_{j\in J} L^2(E_n\cap Y_j,\mu)$$

and this isomorphism is as representations of $L^\infty(Y,\mu)$, working on each direct summand we may assume that $n(x)$ is constant, order J we assume that $J=\mathbb{N}$. In this case set

$$X = \bigcup_{j=1}^{\infty} E_j$$

$$n_k(x) = \sum_{j=1}^{k} \chi_{E_j}(x)$$

$$A_{jk} = \{x : n_k(x) = j\}, j \le k.$$

Define a linear map $T_k\colon L^2(E_k,\mu)\to L^2(X,\mathbb{C}^n,\mu)$ by

$$T_k f(x) = \sum_{j=1}^{k} \chi_{A_{jk}(x)} f(x) e_j.$$

Then T_k intertwines the actions of $L^\infty(X,\mu)$ on $L^2(E_k,\mu)$ and $L^2(X,\mathbb{C}^n,\mu)$ and

$$\bigoplus_{k=1}^{\infty} T_k\colon \bigoplus_{k=1}^{\infty} L^2(E_k,\mu) \to L^2(X,\mathbb{C}^n,\mu)$$

gives a unitary isomorphism between the two representations.

□

2.2. **Direct Integrals of Operators and Representations.** In this section, we study fields of operators acting on measurable fields of Hilbert spaces. We state necessary and sufficient conditions for an operator acting on a direct integral of Hilbert spaces to come from a measurable field of operators. We also show when we can decompose a representation of a separable C^*-algebra as a direct integral of representations.

Definition 2.2.1. Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y and let $\{H_y : y \in Y\}$ be a measurable field of Hilbert spaces over Y. On $\int_Y H_y\,d\mu(y)$ define a representation ρ of $L^\infty(X,\mu)$ by $\rho(f)\xi(x) = f(x)\xi(x)$. We call the image of this representation *the diagonal algebra.*

Definition 2.2.2. *emphLet μ be a Radon measure on a locally compact Hausdorff space Y and let $\{H_x : x \in X\}$ be a measurable field of Hilbert spaces over Y. By a measurable field $\{T_y : y \in Y\}$*of operators we mean an element $(T_y)_{y\in Y} \in \prod_{y\in Y} B(H_y)$ such that

(i) $y \to \langle T_y\xi(y), \eta(y)\rangle$ is measurable for all $\xi, \eta \in \mathcal{S}$. (ii) The essential supremum of $\|T_y\|$ is finite.

Note that

$$\|T_y\| = \sup_{n,m} |\langle T_y\xi_n(y), \xi_m(y)\rangle|$$

for $\xi_n \in \mathcal{S}$ such that $H_y = \overline{\mathrm{Span}\{\xi_n(x) : n \in \mathbb{N}\}}$ almost everywhere, thus it makes sense to ask that the essential supremum is finite.

If $\{T_y : y \in Y\}$ is a measurable field of operators, then we can define an operator T on $\int_Y H_y \, d\mu(y)$ by

$$(T\xi)(y) = T_y\xi(y)$$

we shall denote this operator by $\int_Y^{\oplus} T_y \, d\mu(y)$.

Definition 2.2.3. Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y and let $\{H_y : y \in Y\}$ be a measurable field of Hilbert spaces over Y. Let T be an operator on $\int_Y H_y \, d\mu(y)$ we say that T is *decomposable* if there exists a measurable field $(T_y)_{y\in Y}$ of operators such that

$$T = \int_Y^{\oplus} T_y \, d\mu(y).$$

Direct computations show that following algebraic formulas:

$$\int_Y^{\oplus} T_y \, d\mu(y) + \int_Y^{\oplus} S_y \, d\mu(y) = \int_Y^{\oplus} T_y + S_y \, d\mu(y)$$

$$\left(\int_Y^{\oplus} T_y \, d\mu(y)\right)\left(\int_Y^{\oplus} S_y \, d\mu(y)\right) = \int_Y^{\oplus} T_y S_y \, d\mu(y)$$

$$\left(\int_Y^{\oplus} T_y \, d\mu(y)\right)^* = \int_Y^{\oplus} T_y^* \, d\mu(y).$$

Our first Theorem gives an alternative characterization of decomposable operators.

Theorem 2.2.4. *Let μ be a Radon measure on a locally compact Hausdorff space Y and let $\{H_y : y \in Y\}$ be a measurable field of Hilbert spaces over Y, and set $H = \int_Y H_y \, d\mu(y)$. Let $T \in B(H)$, a necessary and sufficient condition that T be decomposable is that it commutes with the diagonal algebra. If T is decomposable and $T = \int_Y^{\oplus} T_y \, d\mu(y) = \int_Y S_y \, d\mu(y)$, are two decompositions of T, then $S_y = T_y$ almost everywhere. Further $\|T\|$ is the essential supremum of $\|T_y\|$.*

Proof. If T is decomposable it clearly commutes with the diagonal algebra. Conversely, suppose that T commutes with the diagonal algebra and let $(\xi_n(y))$ be sequence in $\mathcal{S}$ such that H_y is the closed linear span on $\xi_n(x)$ for almost every x. Set $\eta_n(y) = (T\xi_n)(y)$, we claim that for all $c_1, \ldots, c_m \in \mathbb{Q}[i] = \{a + bi : a, b \in \mathbb{Q}\}$

$$\left\|\sum_{j=1}^m c_j\eta_n(y)\right\| \leq \|T\| \left\|\sum_{j=1}^m c_j\xi_n(y)\right\|$$

for almost every y. To see this, it suffices to show that for all $f \in L^{\infty}(Y,\mu)\cap L^2(X,\mu)$ we have

$$\int_Y |f(y)|^2 \left\|\sum_{j=1}^m c_j\eta_n(y)\right\|^2 \,d\mu(y) \leq \|T\|^2 \int_Y |f(y)|^2 \left\|\sum_{j=1}^m c_j\xi_n\right\|^2 \,d\mu(y).$$

We have, using $\rho(f)\xi(y) = f(y)\xi(y)$ for $f \in L^{\infty}(Y,\mu)$, that

$$\int_Y |f(y)|^2 \left\|\sum_{j=1}^m c_j\eta_n(y)\right\|^2 \,d\mu(y) = \int_Y \left\|\rho(f)T\left(\sum_{j=1}^m c_j\xi_j\right)(y)\right\|^2 \,d\mu(y) =$$

$$\left\|\rho(f)T\left(\sum_{j=1} c_j\xi_j\right)\right\|^2 = \left\|T\rho(f)\left(\sum_{j=1}^m c_j\xi_j\right)\right\|^2 \leq \|T\|^2 \left|\rho(f)\left(\sum_{j=1}^m c_j\xi_j\right)\right\|^2$$

$$= \|T\|^2 \int_Y |f(y)|^2 \left\|\sum_{j=1}^m c_j\xi_j(y)\right\|^2 \,d\mu(y).$$

Because there are countably many vectors of the form $\sum_{j=1}^m c_j\xi_j$ with $c_j \in \mathbb{Q}[i]$, we may select a conull $Y_0 \subseteq Y$ such that for all $y \in Y_0$, and all $c_1,\dots,c_m \in \mathbb{Q}[i]$ we have

$$T\left(\sum_{j=1}^m c_j\xi_j\right)(y) = \sum_{j=1}^m c_j\eta_j(y)$$

and

$$\left\|\sum_{j=1}^m c_j\eta_j(y)\right\| \leq \|T\| \left\|\sum_{j=1}^m c_j\xi_j(y)\right\|.$$

For $y \in Y_0$ define

$$T_y\left(\sum_{j=1}^m c_j\eta_j(y)\right) = \sum_{j=1}^m c_j\xi_j(y),$$

the above inequality guarantees that T_y is well-defined and $\mathbb{Q}[i]$-linear. It also shows that

$$\|T_y(\xi-\eta)\| \leq \|T\|\|\xi-\eta\|$$

for ξ,η in the $\mathbb{Q}[i]$-span of $\xi_n(y)$. Thus T_y is uniformly continuous, and hence by completeness of H_y, it has a unique extension $T_y\colon H_y \to H_y$, it is easy to check that this extension is linear and has norm at most $\|T\|$ on Y_0. A simple density argument shows that $T = \int_Y^{\oplus} T_y\,d\mu(y)$.

Suppose $T = \int_Y^{\oplus} S_y\,d\mu(y)$ is another decomposition of T, we must show that $S_y = T_y$ almost everywhere. Taking differences, we may assume that $T = 0$ and show that $S_y = 0$ almost everywhere. But for each n, we have that

$$0 = \|T\xi_n\|^2 = \int_Y \|S_y\xi_n(y)\|^2\,d\mu(y)$$

and thus we may select a conull subset on which $S_y\xi_n(y) = 0$ for all n. Because the $\xi_n(y)$ generate S_y, this shows that $S_y = 0$ almost everywhere.

For the last claim, note that if T is a decomposable operator, then since we already checked uniqueness, we can use any decomposition $\int_Y^{\oplus} T_y\,d\mu(y)$ to show that $\|T\|$ is the essential supremum of $\|T_y\|$. The decomposition we constructed in the first part of the proof had $\|T_y\| \leq \|T\|$ almost everywhere by construction. For the reverse inequality simply note that if the essential supremum of $\|T_y\|$ is at most R then for all $\xi \in \mathcal{S}$,

$$\|T\xi\|^2 = \int_Y \|T_y \xi(y)\|^2 \, d\mu(y) \leq R^2 \int_Y \|\xi(y)\|^2 \, d\mu(y) = R^2 \|\xi\|^2.$$

Thus $\|T\|$ is at most the essential supremum of $\|T_y\|$ and this completes the proof. □

Proposition 2.2.5. *Let μ be a Radon measure on a locally compact Hausdorff space Y and let $\{H_y : y \in Y\}$ be a measurable field of Hilbert spaces over Y. Let $T_n = \int_Y^{\oplus} T_n(y) \, d\mu(y)$ be a sequence of decomposable operators. If $T_n \to T$ in the strong operator topology, then T is decomposable. If we write $T = \int_Y^{\oplus} T_y \, d\mu(y)$, then there is a subsequence T_{n_k} such that $T_{n_k}(y) \to T(y)$ is strong operator topology for almost every y.* [3]

Proof. The fact that T is decomposable is obvious from the above proposition. To see the second claim, let (ξ_n) be a dense sequence in H. By a diagonal argument and Theorem 1.1.8 we can find a subsequence T_{n_k} of T_n such that $T_{n_k}(y)\xi_n \to T(y)\xi_n$ in norm for almost every y. By the principal of uniform boundedness, we can find an $M > 0$ such that $\|T_n\| \leq M$ for every n. The above proposition implies that $\|T_n(y)\| \leq M$ for almost every y and all n. Therefore, by removing a set of measure zero we may assume that $\|T_{n_k}(y)\| \leq M$ for all k and $y \in Y$. Since ξ_n is dense this implies that $T_{n_k}(y)\xi \to T(y)\xi$ in norm for every $\xi \in H$. □

We now consider direct integrals of representations.

Definition 2.2.6. Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, and let $\{H_x : x \in X\}$ be a measurable field of Hilbert spaces over Y. Let A be a C^*-algebra, a *measurable field of representations* $\{\pi_y : y \in Y\}$ of A is a collection of $*$-representations $\pi_y \colon A \to B(H_y)$, for $y \in Y$, non-degenerate $*$-representations such that $(y \to \pi_y(a)\xi(y)) \in \mathcal{S}$ for all $\xi \in \mathcal{S}$. Define $\pi \colon A \to B\left(\int_Y^{\oplus} H_y \, d\mu(y)\right)$ by $(\pi(a)\xi)(y) = \pi_y(a)\xi(y)$. Then π is a $*$-representation of A, which we call *the direct integral of* π_y and denote $\pi = \int_Y^{\oplus} \pi_y \, d\mu(y)$.

As in the case of decomposable operators, we have an abstract characterization of direct integrals of representations.

Theorem 2.2.7. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, and let $\{H_x : x \in X\}$ be a measurable field of Hilbert spaces over Y. Set $H = \int_Y^{\oplus} H_y \, d\mu(y)$, suppose $\pi \colon A \to B(H)$ is a $*$-representation such that $\pi(x)$ commutes with the diagonal algebra for all $x \in X$. Then there is a measurable field $\{\pi_y : y \in Y\}$ of $*$-representations of A, such that*

$$\pi = \int_Y^{\oplus} \pi_y \, d\mu(y).$$

[3]The same can be said about any of the usual operator topologies we typically consider, and the proof given below works for any of them.

Proof. Fix a countable subring $R \subseteq A$, which is norm dense, closed under adjoints and scaling by elements in $\mathbb{Q}[i]$. For each $a \in R$, by Theorem 2.2.4 we can find a measurable field of operators a_y such that

$$\pi(a) = \int_Y^{\oplus} a_y \, d\mu(y)$$

and we have, for almost every $y \in Y$ that $\|a_y\| \leq \|\pi(a)\| \leq \|a\|$. For $a \in R$ set

$$N_a^1 = \{y \in Y : \|a_y\| > \|\pi(a)\|\}$$
$$N_a^2 = \{y \in Y : a_y \neq a_y^*\}$$

and for $a, b \in R, \lambda \in \mathbb{Q}[i]$ set

$$N_{ab}^1 = \{y \in Y : a_y b_y \neq (ab)_y\}$$
$$N_{ab}^2 = \{y \in Y : a_y + b_y \neq (a+b)_y\}$$
$$N_{\lambda a} = \{y \in Y : \lambda a_y \neq (\lambda a)_y\}$$

then each of $N_a^1, N_a^2 N_{ab}^1, N_{ab}^2, N\lambda a$ are all null sets. Since R is countable

$$N = \bigcup_{a \in R} \left(N_a^1 \cup N_a^2\right) \cup \bigcup_{a,b \in R} \left(N_{ab}^1 \cup N_{ab}^2\right) \cup \bigcup_{\lambda \in \mathbb{Q}[i], a \in R} N_{\lambda a}$$

is a null set, set $Y_0 = Y \setminus N_0$. For $y \in Y_0$, we have that

$$\pi_y(a) = a_y$$

is a $*$-homomorphism on R and $\|\pi_y(a) - \pi_y(b)\| \leq \|\pi(a-b)\| \leq \|a-b\|$ on Y_0. Thus π_y is uniformly continuous on R for $y \in Y_0$, and since $B(H_y)$ is complete we have a uniqueness extension $\pi_y \colon A \to B(H_y)$ for $y \in Y_0$. Simple density arguments show that π_y are $*$-homomorphisms such that

$$\pi = \int_Y^{\oplus} \pi_y \, d\mu(y).$$

□

Lemma 2.2.8. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, and let H be a fixed separable Hilbert space. let A be a separable $C*$-algebra and $\{\pi_y\}$ a measurable field of representations on H_y.*[4] *Then there are representations $(\rho_y)_{y \in Y}$ such that $y \to \rho_y(a)$ is strongly* Borel for each $y \in Y$, and such that $\pi_y = \rho_y$ almost everywhere.*

Proof. Let $R \subseteq A$ be a countable dense subring closed under the $*$ operation and scaling by elements in $\mathbb{Q}[i]$. By Corollary 1.2.4, for each $a \in R$ we can find $T_a \colon Y \to B(H)$ strongly* Borel such that $T_a(y) = \pi_y(a)$ almost everywhere. As in the previous proposition, we can find a Borel null set $N \subseteq Y$ such that for $y \in Y \setminus N$ we have

$$T_{ab}(y) = T_a(y) T_b(y) \text{ for all } a, b \in R$$
$$T_a(y)^* = T_{a^*}(y) \text{ for all } a \in R$$
$$T_{a+b}(y) = T_a(y) + T_b(y) \text{ for all } a, b \in R$$
$$T_{\lambda a}(y) = \lambda T_a(y) \text{ for all } a \in R, \lambda \in \mathbb{Q}[i]$$
$$\|T_a(y)\| \leq \|a\| \text{ for all } a \in R.$$

[4]This just means $y \to \pi_y(a)$ is strongly* μ-measurable for every $a \in A$.

As in the previous proposition for $y \in Y \setminus N$, the map $y \to T_a(y)$ extends uniquely to a $*$-representation of A, which we denote by ρ_y. By definition $y \to \chi_{Y \setminus N}\rho_y$ is strongly* Borel for each $a \in R$, and since being Borel is closed under norm limits (see Proposition 1.1.2) it follows that $y \to \rho_y(a)$ is strongly*-Borel for each $a \in A$. A similar argument shows that we can find a null set N' such that $\rho_y(a) = \pi_y(a)$ for all $a \in A$.

□

Theorem 2.2.9. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let $\{H_y : y \in Y\}$ be a measurable field of Hilbert spaces over Y. Let A be a seperable $C*$-algebra and π_y, ρ_y are two measurable fields of representations on H_y such that $\pi_y \cong \rho_y$ for almost every $y \in Y$. Then, for almost every $y \in Y$, we can find a unitary $U_y \in B(H_y)$ such that $y \to U_y\xi(y) \in \mathcal{S}, y \to U(y)^*\xi(y) \in \mathcal{S}$ for each $\xi \in \mathcal{S}$ such that $U_y\pi_yU_y^* = \rho_y$. Thus $\int_Y^{\oplus} U_y\, d\mu(y)$ implements a unitary equivalence between $\int_Y^{\oplus} \pi_y \cong \int_Y^{\oplus} \rho_y$.*

Proof. Let $Y_n' = \{y \in Y : \dim H_y = n\}$, for $n \in \mathbb{N} \cup \{\infty\}$, and choose $Y_n \subseteq Y_n'$ Borel such that $\mu(Y_n' \setminus Y_n) = 0$. Working on each Y_n, we may instead assume that Y is a Borel subset of a second countable locally compact Hausdorff space and that there is a fixed separable Hilbert space H, such that $\{\pi_y : y \in Y\}$ is a measurable field of representations on H. Observe that the one-point compactification of a second countable locally compact Hausdorff space is still second countable, and being compact, it is metrizable. In particular a second countable locally compact Hausdorff space is homeomorphic to an open subset of a Polish space, and is thus Polish by Theorem 4.1.13. Since $\mathcal{U}(H)$ is closed in $\{T \in B(H) : \|T\| = 1\}$ for the strong* topology, it is a Polish space by Proposition 1.2.6. By applying the previous lemma, we may assume that $y \to \pi_y(a), y \to \rho_y(a)$ are strongly* Borel for each $a \in A$. Let

$$X = \{(y, U) \in Y \times \mathcal{U}(H) : U\pi_y(a)U^* = \rho_y(a) \text{ for all } a \in A\}.$$

We claim that X is a Borel subset of $Y \times \mathcal{U}(H)$, when $\mathcal{U}(H)$ is given the strong* topology. Indeed, let D be a countable dense subset of A, then

$$X = \bigcap_{a \in D} \{(y, U) \in Y \times \mathcal{U}(H) : U\pi_y(a)U^* = \rho_y(a)\},$$

so it suffices to show that for $S, T \colon Y \to B(H)$ strongly* Borel

$$\{(y, U) \in Y \times \mathcal{U}(H) : UT(y)U^* = S(y)\}$$

is Borel. But if $(\xi_n)_{n=1}^{\infty}$ is a dense sequence in H, then

$$\{(y, U) \in Y \times \mathcal{U}(H) : UT(y)U^* = S(y)\} = \bigcap_{n=1}^{\infty} \{(y, U) \in Y \times \mathcal{U}(H) : UT(y)U^*\xi_n = S(y)\xi_n\}.$$

Since $y \to UT(y)U^*$ is strongly * Borel by Propositions 1.2.6, we have that

$$\{(y, U) \in Y \times \mathcal{U}(H) : UT(y)U^* = S(y)\}$$

is a Borel set. Let $\pi_1 \colon X \to Y, \pi_2 \colon X \to \mathcal{U}(H)$ be the projections onto the first coordinate and second coordinates, our assumptions imply that $\pi_1(X)$ is a conull set in Y, and it is analytic by Corollary 4.1.22. Since X is a Borel subset of the Polish space $Y \times \mathcal{U}(H)$, Theorem 4.1.33 implies that we can find a universally measurable

$\phi\colon Y \to X$ such that $\pi \circ \phi(y) = y$ for all $y \in \pi(X)$. Let $U\colon \pi(X) \to \mathcal{U}(H)$ be defined by $U = \pi_2 \circ \phi$. Since

$$\phi(y) = (y, U(y)) \in Y \text{ for all } y \in \pi_1(X)$$

we have

$$U(y)\pi_y(a)U(y)^* = \rho_y(a)$$

for each $y \in \pi_1(X)$, and $\pi_1(X)$ was already seen to be conull. Thus $U(y)$ has the desired properties.

□

3. Direct Integrals of Von Neumann Algebras

3.1. **Measurable Fields of Von Neumann Algebras and Decomposition Into Factors.** In the previous sections, we have discussed the notions of a measurable field of Hilbert Spaces, now we shall consider a measurable field of von Neumann algebras acting on a measurable field of Hilbert spaces.

Definition 3.1.1. Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, and let $\{H_y : y \in Y\}$ be measurable field of Hilbert spaces over Y. A collection of von Neumann algebras $\{M_y : y \in Y\}$ with $M_y \subseteq B(H_y)$, such that the identity of M_y is the identity of $B(H_y)$ for almost every y is said to be *a measurable field of von Neumann algebras* if there are measurable fields of operators $\{a_n(y) : y \in Y\}$ such that $M_y = W^*(\{a_n(y) : n \in \mathbb{N}\})$ for almost every y. If $H_y = H$ is a fixed separable Hilbert space for each y, we say that M_y is a *Borel field of von Neumann algebras* if the a_n above can be chosen to be strongly* Borel for each n, and $M_y = W^*(\{a_n(y) : y \in Y\})$ for every $y \in Y$.

We have the following consequence of Corollary 1.2.4. If $y \to M_y$ is a measurable field of von Neumann algebras acting on H, then there is a Borel field $y \to \widetilde{M_y}$ of von Neumann algebras acting on H such that $\widetilde{M_y} = M_y$ for almost every y. We shall often say "let $(M_y, H_y)_{y\in Y}$ be a measurable field of von Neumann algebras over Y to indicate that H_y is a measurable field of Hilbert spaces over Y, and that M_y is a measurable field of von Neumann algebras such that $M_y \subseteq B(H_y)$.

Our first propositions of this section establishes basic facts about measurable and Borel fields of von Neumann algebras.

Proposition 3.1.2. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, let H be a separable Hilbert space and $(M_y)_{y\in Y}$ a Borel field of von Neumann algebras acting on H. Then*

(a) $(M_y')_{y\in Y}$ is a measurable field of von Neumann algebras.

(b) If N_y is another Borel field of von Neumann acting on H, then $N_y \cap M_y$ and $N_y \vee M_y$ are measurable fields of von Neumann algebras.

(c) Fix $R > 0$, then using $\overline{B(0,R)} = \{T \in B(H) : \|T\| \leq R\}$ we have that

$$\{(y,T) \in Y \times \overline{B(0,R)} : T \in M_y\}$$

is a Borel subset of $Y \times \overline{B(0,R)}$.

(d) The set

$$\{(y,\phi) \in Y \times B(H)_* : \phi\big|_{M_y} \text{ is a trace}\}^5$$

is a Borel subset of $Y \times B(H)_*$.

[5]Here we are identifying an element in $B(H)_*$ with the linear functional on $B(H)$ it defines

(e) Let A be a separable C^*-algebra and $\{\pi_y : y \in Y\}$ a measurable field of representations over Y. Then π_y'' is a measurable field of von Neumann algebras.

Proof. Let $y \to a_n(y)$ be strongly* Borel maps such that $M_y = W^*(\{a_n(y) : n \in \mathbb{N}\})$ for every $y \in Y$.

(a) Let ϕ_n be a dense sequence in $B(H)_*$. The idea is to produce a sequence $b_n : Y \to B(H)$ of strongly* μ-measurable maps such that

(i) $\|b_n(y)\| \leq 1$ for all y

(ii) $\|\phi_n(b_n(y))\| \geq \frac{\|\phi_n\big|_{M_y}\|}{2}$

(iii) $b_n(y) \in M_y$ for all y.

Once we have done this, the Hahn-Banach Theorem guarantees that $M_y' = W^*(\{b_n(y) : n \in \mathbb{N}\})$ for every y, and thus that M_y' is a measurable field of von Neumann algebras.

To do this, we first show that for all $\phi \in B(H)_*$ we have $y \to \|\phi\big|_{M_y}'\|$ is Borel. Let $f_y : B(H) \to l^\infty(\mathbb{N})\overline{\otimes}B(H)$ be defined by $f_y(T) = (Ta_n(y) - a_n(y)T)_{n\in\mathbb{N}}$, then $M_y' = \ker(\psi_y)$, and ψ_y is weak* continuous. Thus $(M_y')^\perp = \overline{f_y^t((l^\infty(\mathbb{N})\overline{\otimes}B(H))_*)}$. We can identify $(l^\infty(\mathbb{N})\overline{\otimes}B(H))_*$ as all $l^1(\mathbb{N}, B(H)_*)$. Let $\psi^{(j)} = (\psi_n^{(j)})_{n=1}^\infty$ be a dense sequence in $l^1(\mathbb{N}, B(H)_*)$. Then

$$\|\phi\big|_{M_y}\| = \inf\{\|\phi - \psi\| : \psi \in (M_y')^\perp\} =$$

$$\inf\{\|\phi - f_y(\psi^{(j)}\| : j \in \mathbb{N}\}.$$

Let T_k be a weak* dense sequence in the norm closed unit ball of $B(H)$. Because

$$f_y^t(\psi^{(j)})(T) = \sum_{n=1}^\infty \phi_n^{(j)}(T_k a_n(y) - a_n(y)T_k)$$

we have that

$$\|\phi - f_y(\psi^{(j)}\| = \sup_k \left\|\phi(T_k) - \sum_{n=1}^\infty \phi_n^{(j)}(T_k a_n(y) - a_n(y)T_k)\right\|.$$

Because $a_n(y)$ are strongly* Borel, it follows from this formula and Proposition 1.1.2 that $\|\phi - f_y(\psi^{(j)}\|$ is Borel for all j. Thus $\|\phi\big|_{M_y}\|$ is Borel.

We have that

$$M_y' = \{T \in B(H) : Ta_n(y) = a_n(y)T\}$$

for every $y \in Y$. Note that by Proposition 1.2.6

$$\bigcap_{n=1}^\infty \{(T, y) \in B(H) \times Y : \|T\| \leq 1, Ta_n(y) = a_n(y)T\}$$

is a Borel subset of $\overline{B(0,1)} \times Y$. This set is precisely $\{(T, y) \in B(H) : \|T\| \leq 1, T \in M_y'\}$. By a similar logic

$$X_n = \{(T, y) \in B(H) \times Y \times B(H)_* : T \in M_y', \|T\| \leq 1, \|\phi_n(T)\| \geq \frac{\|\phi_n\big|_{M_y}\|}{2}\}$$

is Borel and if $\pi : X_n to Y$ is the projection , it is easy to see that π is onto. Thus by Theorem 4.1.33 we can find a universally measurable $\phi_n : Y \to X_n$ such that $\pi \circ \phi_n = \mathrm{Id}$. Setting $b_n = q \circ \phi_n$ where q is projection onto the second coordinate, we see that the b_n have the desired properties.

(b) It is clear from the definitions that $M_y \vee N_y$ is a Borel field of von Neumann algebras, since

$$N_y \cap M_y = (M_y \cap N_y)'$$

the other claim follows from (a). (c) We may assume that the $a_n(y)$ are self-adjoint for each y. Fix a dense sequence ξ_k in H. Let A be the ring of all noncommutative polynomials in variables $x_1, x_2, \ldots$ with coefficents in $\mathbb{Q}[i]$. Then A is countable, we claim that

$$\{(T, y) \in B(H) \times Y : \|T\| \leq R, T \in M_y\} =$$
$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{P \in A} E_P.$$

Where

$$E_P =$$
$$\bigcap_{j=1}^{m} \{(T, y) : \|T\| \leq R, \|P(a_1(y), \ldots)\xi_k - T\xi_k\| < 1/n\} \cap \{(T, y) : \|P(a_1(y), \ldots)\| \leq 2R+1\}.$$

Indeed if (T, y) is in the right hand-side, then that means we can find a net $P_i \in A$, such that

$$P_i(a_1(y), a_2(y), \ldots)\xi_k \to T\xi_k$$

for all k, and $\|P_i(a_1(y), \ldots)\| \leq 2R+1$. Since $\|P_i(a_1(y), \ldots)\| \leq 2R+1$, the density of the ξ_k guarantees that

$$P_i(a_1(y), a_2(y), \ldots) \to T$$

strongly and T is in M_y. Conversely, if $T \in M_y$, then by Kaplansky's Density Theorem given any $n, k \in \mathbb{N}$ we can find a noncommutative polynomial P with coefficents in $\mathbb{C}$, in the variables $x_1, x_2, \ldots$ such that

$$\|P(a_1(y), a_2(y), \ldots)\xi_j - T\xi_j\| < 1/n \text{ for } j = 1, \ldots, k$$

and

$$\|P(a_1(y), a_2(y), \ldots)\| \leq R.$$

Perturbing the coefficents slightly we may find $P' \in A$ such that

$$\|P(a_1(y), a_2(y), \ldots)\xi_j - T\xi_k\| < 1/n \text{ for } j = 1, \ldots, k$$
$$\|P(a_1(y), a_2(y), \ldots)\| \leq 2R + 1.$$

Since E_P is Borel by Proposition 1.2.6 we have that

$$\{(T, y) \in B(H) \times Y : \|T\| \leq 1, T \in M_y\}$$

is Borel.

(d) Adding in all $P(a_1(y), a_2(y), \ldots)$ where P is in the ring A define in (c), we may assume that the linear span of $a_n(y)$ is weak* dense in M_y for all y. Then

$$\{(\phi, y) \in B(H)_* \times Y : T\big|_{M_y} \text{ is a trace }\} =$$
$$\bigcup_{n,m \in \mathbb{N}} \{(\phi, y) \in B(H)_* \times Y : \phi(a_m(y)a_n(y)) = \phi(a_n(y)a_m(y))\},$$

and this is a Borel set by Proposition 1.2.6.

(e) If $(x_n)_{n=1}^{\infty}$ is a dense sequence in A, then $\{\pi_y(a_n) : n \in \mathbb{N}\}$ generates $\pi_y(A)''$ for every y.

$\square$

Corollary 3.1.3. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, let $(M_y, H_y)_{y\in Y}$ be a measurable field of von Neumann algebras over Y. Then*

(a) $(M'_y)_{y\in Y}$ is a measurable field of von Neumann algebras.

(b) If N_y is another measurable field of von Neumann, then $N_y \cap M_y$ and $N_y \vee M_y$ are measurable fields of von Neumann algebras.

Proof. By partitioning and applying Theorem 2.1.9 we may assume that $H_y = H$ is constant. Now simply modify M_y, N_y on null sets to make them Borel and apply the previous proposition.

□

Definition 3.1.4. Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y, and let $(M_y, H_y)_{y\in Y}$ be a measurable field of von Neumann algebras over Y. Let M be the set of all operators $\int_Y^{\oplus} a_y \, d\mu(y)$ such that $a_y \in M_y$ for almost every $y \in Y$. *Then M is called the direct integral of M_y* and denoted $\int_Y^{\oplus} M_y \, d\mu(y)$.

Theorem 3.1.5. *Let μ be a Radon measure on a σ-compact locally compact Hausdorff space Y.*

(a) Let $(M_y, H_y)_{y\in Y}$ be a measurable field of von Neumann algebras over Y. Then $M = \int_Y^{\oplus} M_y \, d\mu(y)$ is a von Neumann algebra and

$$M' = \int_Y^{\oplus} M'_y \, d\mu(y)$$

(b) Let A be a separable C^*-algebra, and let $\{\pi_y : y \in Y\}$ be a measurable field of representations over Y, such that π_y is non-degenerate for almost every y, and set $\pi = \int_Y^{\oplus} \pi_y \, d\mu(y)$. Then $\pi(A)'' \subseteq \int_Y^{\oplus} \pi_y(A)'' \, d\mu(y)$.

(c) If $(M_y, H_y), (N_y, H_y)$ are two measurable fields of von Neumann algebras, then

$$\int_Y^{\oplus} N_y \, d\mu(y) \subseteq \int_Y^{\oplus} M_y \, d\mu(y)$$

if and only if $N_y \subseteq M_y$ almost everywhere. Further we have

$$\int_Y^{\oplus} N_y \cap M_y \, d\mu(y) = \int_Y^{\oplus} M_y \, d\mu(y) \cap \int_Y^{\oplus} N_y \, d\mu(y).$$

(d) If $p \in M$ is a projection, and $p = \int_Y^{\oplus} p(y) \, d\mu(y)$ then $pMp = \int_Y^{\oplus} p(y) M_y p(y) \, d\mu(y)$.

Proof. (a)It suffices to show that

$$M' = \int_Y^{\oplus} M'_y \, d\mu(y),$$

because if we take commutants again (and apply the above to M'_y in place of M_y) we see that

$$M'' = \int_Y^{\oplus} M_y \, d\mu(y) = M$$

i.e. M is a von Neumann algebra. It is clear that

$$M' \supseteq \int_Y^{\oplus} M'_y \, d\mu(y).$$

For the converse, let $a_n(y)$ be a measurable field of operators such that $M_y = W^*(a_n(y) : n \in \mathbb{N})$ almost everywhere, and let $T \in M'$. Since the identity operator is in M_y for almost every y, we have that T commutes with the diagonal algebra so that by Theorem 2.2.4 we can write $T = \int_Y^{\oplus} T_y \, d\mu(y)$. Then, for each n and almost every y we have$[T_y, a_n(y)] = 0$. By removing countably many null sets we see that for almost every y we have that $[T_y, a_n(y)] = 0$ for all n.

(b) Since $\pi_y(A)' = (\pi_y(A)'')'$, we have that $\pi_y(A)'$ is a measurable field of von Neumann algebras by the above proposition. Suppose $T \in \pi(A)''$, then we can find $a_n \in A$, such that $\|\pi(a_n)\| \leq \|T\|$ and $\pi(a_n) \to T$ in the strong operator topology. Thus by Theorem 2.2.5 and passing to a subsequence we can write

$$T = \int_Y^{\oplus} T_y \, d\mu(y)$$

in such a way that $\pi(a_n)(y) \to T_y$ in the strong operator topology almost everywhere. Thus

$$T \in \int_Y^{\oplus} \pi_y(A)'' \, d\mu(y).$$

(c) Let $a_n(y), b_n(y)$ be measurable fields of operators generating M_y, N_y almost everywhere. Set

$$N = \int_Y^{\oplus} N_y \, d\mu(y), M = \int_Y^{\oplus} M_y \, d\mu(y).$$

If $N \subseteq M$, then we have that $b_n(y) \in M_y$ for almost every y, thus $N_y \subseteq M_y$ for almost every y. Conversely, suppose that $N_y \subseteq M_y$ for almost every y. We first claim that $M = W^*(a_n : n \in \mathbb{N})$. We have that

$$W^*(a_n : n \in \mathbb{N}) = (\{1\} \cup \{a_n\}_{n=1}^{\infty} \cup \{a_n\})''.$$

But by a similar argument as in (a), we have that

$$(\{1\} \cup \{a_n\}_{n=1}^{\infty} \cup \{a_n\})' = \int_Y^{\oplus} (\{1\} \cup \{a_n\}_{n=1}^{\infty} \cup \{a_n\})' \, d\mu(y)$$

and

$$\int_Y^{\oplus} (\{1\} \cup \{a_n\}_{n=1}^{\infty} \cup \{a_n\})' \, d\mu(y) = \int_Y^{\oplus} M_y' \, d\mu(y)$$

and part (a) now proves the claim, by the double commutant theorem. A similar result holds for N, and since $N_y \subseteq M_y$ we have that $b_n(y) \in M_y$ for almost every y. Thus $b_n \in M$, so

$$N = W^*(\{a_n : n \in \mathbb{N}\}) \subseteq M.$$

For the second claim, note that the logic we just used implies that

$$\int_Y^{\oplus} M_y \vee N_y \, d\mu(y) = \int_Y^{\oplus} M_y \, d\mu(y) \vee \int_Y^{\oplus} N_y \, d\mu(y)$$

and taking commutatns proves the second claim.

(d) It is straightforward from the definition to verify that $y \to p(y)M_y p(y)$ is measurable field of von Neumann algebras. It is clear that

$$\int_Y^{\oplus} p(y)M_y p(y) \, d\mu(y) \subseteq pMp.$$

On the other hand, if $a \in pMp$, then

$$\int_Y^{\oplus} a(y)\, d\mu(y) = a = pap = \int_Y^{\oplus} p(y)a(y)p(y)\, d\mu(y)$$

so that $a(y) = p(y)a(y)p(y)$ for almost every y. That is $a(y) \in p(y)Mp(y)$ for almost every y.

□

Theorem 3.1.6. *Let Y be a σ-compact locally compact Hausdorff space and let μ be a Radon measure on Y. Let (M_y, H_y) be a measurable field of von Neumann algebras over Y, and $M = \int_Y^{\oplus} M_y\, d\mu(y)$.*

(i) If $\phi \in M_*$, then there exists, for almost y, a $\phi_y \in (M_y)_*$ such that $y \to \phi_y(a(y))$ is measurable for all $a \in M$, and

$$\int_Y^{\oplus} \|\phi_y\|\, d\mu(y) < \infty$$

$$\phi(a) = \int_Y \phi_y(a(y))\, d\mu(y)$$

$$\|\phi\| = \int_Y \|\phi_y\|\, d\mu(y).$$

This last formula implies that ϕ_y is unique up to agreement on a null set. Conversely, if $\phi_y \in (M_y)_*$ is such that $y \to \phi_y(a(y))$ is measurable for all $a \in M$, and $\int_Y \|\phi_y\|\, d\mu(y) < \infty$ we have that

$$\phi(a) = \int_Y \phi_y(a(y))\, d\mu(y)$$

is in M_*. We will write $\phi = \int_Y^{\oplus} \phi_y\, d\mu(y)$.

(ii) If ϕ, ϕ_y are as in (i), then $\phi \geq 0$ if and only if $\phi_y \geq 0$ for almost every y.

(iii) $\phi \geq 0$ is faithful if and only if ϕ_y is faithful.

Proof. (i) If $\phi \in M_*$, then by the Hahn-Banch Theorem we can find $\psi \in B(H)_*$ such that $\psi\big|_M = \phi$. Let $\xi_n, \eta_n \in H = \int_Y^{\oplus} H_y\, d\mu(y)$ be such that

$$\sum_{n=1}^{\infty} \|\xi_n\|^2, \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$$

and

$$\psi(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \eta_n\rangle.$$

Then

$$\int_Y \sum_{n=1}^{\infty} \|\xi_n(y)\|^2\, d\mu(y) = \sum_{n=1}^{\infty} \int_Y \|\xi_n(y)\|^2\, d\mu(y) =$$

$$\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty,$$

and similarly for $\eta_n(y)$. Thus

$$\sum_n \|\xi_n(y)\|^2, \sum_n \|\eta_n(y)\|^2 < \infty \text{ for almost every } y..$$

Therefore we can define $\phi_y \in B(H_y)$ by

$$\phi_y(T) = \sum_{n=1}^{\infty} \langle T\xi_n(y), \eta_n(y)\rangle$$

and by definition of a measurable field of operators we have that $y \to \langle a(y)\xi_n(y), \eta_n(y)\rangle$ is measurable for all $a \in M$. Thus we have that $y \to \phi_y(a(y))$ is measurable for all $a \in M$. Further we have that

$$\int_Y \|\phi_y\|\, d\mu(y) \leq \int_Y \sum_{n=1}^{\infty} \|\xi_n(y)\|\|\eta_n(y)\|\, d\mu(y) =$$

$$\sum_{n=1}^{\infty} \int_Y \|\xi_n(y)\|\|\eta_n(y)\|\, d\mu(y) \leq \sum_{n=1}^{\infty} \left(\int_Y \|\xi_n(y)\|^2\, d\mu(y)\right)^{1/2} \left(\int_Y \|\eta_n(y)\|^2\, d\mu(y)\right)^{1/2} \leq$$

$$\left(\sum_{n=1}^{\infty} \|\xi_n\|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|\eta_n\|^2\right)^{1/2} < \infty.$$

A straightforward computation verifies that

$$\int_Y \phi_y(a(y))\, d\mu(y) = \phi(a) \text{ for all } a \in M.$$

If $b_n \in M$ is such that $b_n(y)$ is weak* dense for almost every y, [6] then

$$\|\phi_y\| = \sup_n |\phi_y(b_n(y)|$$

and is thus measurable. Straightfoward estimates show that

$$\|\phi\| \leq \int_Y \|\phi_y\|\, d\mu(y).$$

Since M is the dual of M_*, we can find $a \in M$ with $\|a\| \leq 1$ such that

$$\phi(a) = \|\phi\|.$$

Let $\alpha\colon Y \to \mathbb{T}$ be measurable such that $\alpha\phi_y(a_y) = |\phi_y(a_y)|$ for every y. Then

$$\|\phi\| = \phi(a) = \int_Y \phi_y(a_y)\, d\mu(y) \leq \int_Y |\phi_y(a(y))|\, d\mu(y) =$$

$$\int_Y \phi_y(\alpha(y)a(y))\, d\mu(y) = \phi(\alpha a) \leq \|\phi\|.$$

Thus by replacing a with αa we may assume that $\phi_y(a(y)) \geq 0$ almost everywhere. We claim that $\phi_y(a_y) = \|\phi_y\|$ for almost every y. By applying Theorem 2.1.9 we may assume that there is a partition Y_n of Y and separable Hilbert spaces K_n such that $H_y = K_n$ for $y \in Y_n$. Then by modifying a_y, ϕ_y on a null set we may assume that $y \to a(y), y \to \phi_y$ are strongly* Borel, and Borel, respectively for $y \in Y_n$. For each n, let

$$E_n = \{(y, b) \in Y_n \times B(H) : \|b\| \leq 1, \phi_y(b) > \|\phi_y\|, b \in M_y\}.$$

Then E_n is a Borel set when $B(H) \cap \overline{B(0,1)}$ is given the strong* topology by Proposition 3.1.2. Let $\pi_1\colon E_n \to Y_n, \pi_2\colon E_n \to B(H)$ be the projections. We have that $\pi_1(E_n)$ is analytic, and by Theorem 4.1.33 we can find a universally measurable

[6] e.g. consider all $P(a_1(y), a_2(y), \ldots)$ where P is a noncommutative polynomial with coefficients in $\mathbb{Q}[i]$.

$g\colon Y_n \to E_n$ such that $\pi_1 \circ g = \mathrm{Id}$. Let $b' = \pi_2 \circ g$, , set $b = \chi_{\pi_1(E_n} b' + \chi_{\pi_1(E_n)^c} a$, then $b \in M$. Also

$$\|\phi\| \geq \phi(b) = \int_{\pi_1(E_n} \phi_y(b(y)), d\mu(y) + \int_{\pi_1(E_n)^c} \phi_y(a(y))\, d\mu(y) \geq$$

$$\int_{\pi_1(E_n} \phi_y(b(y)), d\mu(y) + \int_{\pi_1(E_n)^c} \phi_y(a(y))\, d\mu(y) = \phi(a) = \|\phi\|.$$

This implies that $\mu(\pi_1(E_n)) = 0$ for all n. Since $\pi_1(E_n) = \{y : \phi_y(a(y)) < \|\phi_y\|\}$ we have that $\|\phi_y\| = \phi_y(a(y))$ almost everywhere. Thus

$$\phi\| = \int_Y \phi_y(a_y) = \int_Y \|\phi_y\|\, d\mu(y),$$

as desired.

(ii) For the second claim, it is clear that if $\phi_y \geq 0$ almost everywhere then $\phi \geq 0$. Conversely, the equality that we just established

$$\|\phi\| = \int_Y \|\phi_y\|\, d\mu(y)$$

shows that any two decompositions of ϕ agree almost everywhere. Thus if $\phi \geq 0$, then we can restart the construction by extending ψ to a positive normal functional on $B(H)$ and finding $\zeta_n \in H$ such that

$$\psi(T) = \sum_{n=1}^{\infty} \langle T\xi_n, \xi_n \rangle.$$

Exactly as in the first step of the proof we see that $\phi_y(T) = \sum_n \langle T\zeta_n(y), \zeta_n(y)\rangle$ gives a decomposition of ϕ and clearly $\phi_y \geq 0$ almost everywhere.

(iii) Suppose $\phi \geq 0$ and that ϕ_y is faithful for almost every y. If $a \in M$, $a \geq 0$, then $a(y) \geq 0$ almost everywhere and if $\phi(a) = 0$ we have that

$$0 = \int_Y \phi_y(a(y))\, d\mu(y)$$

since $\phi_y(a(y)) \geq 0$ this implies that $\phi_y(a(y)) = 0$ almost everywhere. Since ϕ_y is faithful almost everywhere we must have that $a(y) = 0$ almost everywhere.

Suppose that ϕ_y is faithful for almost y. If $a_n(y)$ are measurable operator fields which generate M_y almost everywhere then $a_n(y)$ give generating fields in $L^2(M_y, \phi_y)$. Thus $L^2(M_y, \phi_y)$ is a meaurable field of Hilbert spaces over Y, and measurable operator fields in M preserve the measurable sections of $L^2(M_y, \phi_y)$. Thus we have a normal representation π of M on

$$H = \int_Y^{\oplus} L^2(M, \phi_y)\, d\mu(y).$$

Moreover if we use $1 \in H$, for the vector which is the identity of M_y for every y we have that

$$\phi(a) = \langle a1, 1 \rangle$$

so to prove that ϕ is faithful, we only have to show that 1 is cyclic for $\pi(M)'$. By a similar logic as above we have that

$$N = \int_Y^{\oplus} M_y'\, d\mu(y)$$

has a representation ρ on H with image contained in $\pi(M)'$. Suppose $\eta \in H$ is perpendicular to $\rho(N)1$, then for every $b \in N$ and $E \subseteq Y$ measurable we have that

$$0 = \langle \rho(\chi_E b)1, \eta \rangle = \int_E \langle \eta(y), b(y) \rangle \, d\mu(y)$$

thus $\langle \eta(y), b(y) \rangle = 0$ for almost every y. If we let $b_n \in N$ be such that $b_n(y)$ is strong* dense in M_y for almost every y, we have for almost every y that $\langle \eta(y), b_n(y) \rangle = 0$ for all n. Since $b_n(y)$ are dense in $L^2(M, \phi_y)$ by the normality of ϕ_y we have that $\eta(y) = 0$ for almost every y. Thus $\eta = 0$, so $\rho(N)1$ is dense in H. Thus 1 is cyclic for M' and thus ϕ is faithful.

□

Corollary 3.1.7. *Let Y be a second countable locally compact Hausdorff space and let μ be a Radon measure on Y. For a fixed Hilbert space H, and a von Neumann algebra $M \subseteq B(H)$. we have that the map $f \otimes T \to (x \to f(x)T)$ induces an isomorphism*

$$L^\infty(Y, \mu) \otimes M \cong \int_Y^\oplus B(H) \, d\mu(y)$$

In particular, by the results of the above theorem we have the identification

$$(L^\infty(Y, \mu) \otimes M)_* \cong L^1(X, M, \mu).$$

Proof. We can represent both algebras on

$$L^2(Y, \mu) \otimes H \cong L^2(Y, \mu, H) \cong \int_Y^\oplus H \, d\mu(y)$$

and this is compatible with the above identification. Under this identification,

$$L^\infty(Y, \mu) \overline{\otimes} M \subseteq \int_Y^\oplus M \, d\mu(y)$$

since the right hand side is a von Neumann algebra which contains $L^\infty \otimes 1 = \int_Y^\oplus \mathbb{C} \, d\mu(y)$ and $1 \otimes M$. Applying this observation to M' we see that

$$(L^\infty(Y, \mu) \overline{\otimes} M)' = L^\infty(Y, \mu) \overline{\otimes} M' \subseteq \int_Y M' \, d\mu(y) = \left(\int_Y M \, d\mu(y) \right)'$$

and taking commutants in this equation completes the proof.

□

We are in position to prove the main theorem of this section, namely that every von Neumann algebra is a direct integral of factors. Before we do so, we will show that the von Neumann algebra structure of a direct integral does not depend upon the field of Hilbert spaces it is represented on, but only on (almost every) the von Neumann algebras we are integrating. To do this we will need a few preliminary results.

Proposition 3.1.8. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let $(M_y, H_y), (N_y, K_y)$ be two measurable fields of von Neumann algebras over Y, and let*

$$M = \int_Y^\oplus M_y \, d\mu(y), N = \int_Y^\oplus N_y \, d\mu(y).$$

If $\pi_y\colon M_y \to N_y$ are normal $*$-homomorphisms such that for every $a \in M$, we have that $y \to \pi_y(a(y))$ is in N, then $\pi\colon M \to N$ defined by $\pi(b)(y) = \pi_y(a(y))$ is a normal $*$-homomorphism, which we will denote by

$$\int_Y^{\oplus} \pi_y \, d\mu(y).$$

Furthermore π is injective if almost every π_y, and π is surjective if almost every π_y is.

Proof. To prove that π is normal, we have to show that for all $\phi \in N_*$ we have that $\phi \circ \pi \in M_*$. Fix $\phi \in N_*$ and write $\phi = \int_Y^{\oplus} \phi_y \, d\mu(y)$ by Theorem 3.1.6. In order to show that $\phi \circ \pi \in M_*$, we have to show that $\ker(\phi \circ \pi)$ is weak* closed, by the Krein-Smulian Theorem we have to show that $\ker(\phi \circ \pi) \cap \overline{B(0,1)}$ is weak* closed. So we need to show that $\ker(\phi \circ \pi) \cap \overline{B(0,1)}$ is weak operator topology closed. Since M has seperable predual, its weak* topology is metrizable on bounded sets, so we need to show that if $a_n \in \ker(\phi \circ \pi) \cap \overline{B(0,1)}$ and $a_n \to a$ in the weak* topology, then $a \in \ker(\phi \circ \pi) \cap \overline{B(0,1)}$. Note that $\|a\| \leq 1$, so that $\|a(y)\| \leq 1$ almost everywhere, and since π_y is contractive we have that $\|\pi_y(a(y))\| \leq 1$ almost everywhere. Since π_y is weak* continuous for each y, the dominated convergence theorem implies that

$$\phi(a) = \int_Y \phi_y(a(y)) \, d\mu(y) = \lim_{n \to \infty} \int_Y \phi_y a_n(y) \, d\mu(y) = 0.$$

Thus $a \in \ker(\phi \circ \pi) \cap B(0,1)$ and the proof is complete.

For the last two assertions, first note that $\pi_* \phi = \int_Y^{\oplus} (\pi_y)_* \phi_y \, d\mu(y)$ for all $\phi = \int_Y^{\oplus} N_y \, d\mu(y) \in N_*$. Suppose π_y is surjective for almost every y. Since $\pi\colon M \to N$ is a normal *-homomorphism, we have that $\pi(M)$ is a von Neumann algebra, so we need to show that $\pi(M)$ is weak* dense in N. So by the Hahn-Banach theorem we have to show that if $\phi \in N_*$ is such that $\phi = 0$ on $\pi(M)$, then $\phi = 0$. Given $\phi \in M_*$ as such that $\phi = 0$ on M, write $\phi = \int_Y^{\oplus} N_y \, d\mu(y)$, and set $\psi_y = (\pi_y)_* \phi_y$ so that $\psi = \pi_* \phi = 0$, thus

$$0 = \int_Y \|\psi_y\| \, d\mu(y).$$

Thus $\psi_y = 0$ almost everywhere, but since π_y is surjective almost everywhere this implies that $\phi_y = 0$ almost everywhere, i.e. that $\phi = 0$.

The injectivity is easier, since π_y is isometric almost everywhere we have that

$$\|\pi(a)\| = \| \|\pi(a_y)\| \|_{L^\infty(Y)} = \| \|a_y\| \|_{L^\infty(Y)} = \|a\|.$$

□

Theorem 3.1.9. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let $(M_y, H_y), (N_y, K_y)$ be two measurable fields of von Neumann algebras over Y. Set*

$$H = \int_Y^{\oplus} H \, d\mu(y), M = \int_Y^{\oplus} M_y \, d\mu(y)$$

$$K = \int_Y^{\oplus} K \, d\mu(y), N = \int_Y^{\oplus} N_y \, d\mu(y).$$

Suppose that $(M_y, H_y), (N_y, K_y)$ are unitary equivalent for almost every y. Then there are unitary $U_y\colon H_y \to K_y$ for almost every y, such that for all $\xi \in H, \eta \in K$ we have $y \to U_y \xi(y) \in K, y \to U(y)\eta(y) \in H$, and $U_y M_y U_y^* = N_y$ for almost every

y. Thus $U=\int_Y^{\oplus} U_y\,d\mu(y)$ implements a unitary equivalence between (M,H) and (N,K).

Proof. By decomposing into direct sums, we may assume that $y\to M_y, y\to N_y$ are Borel and that $H_y=H, K_y=K$ are fixed separable Hilbert spaces for each y. Let $a_n(y), b_n(y)$ be Borel operator fields which generate M_y, N_y for every y. Let

$$X=\bigcap_{n=1}^{\infty}\{(y,U): U: H\to K \text{ unitary and } U^*b_n(y)U\in N_y, Ua_n(y)U^*\in M_y\}$$

since the unitary operators from H to K are strong*-closed we have by Proposition 3.1.2 that X is a Borel set. So if $\pi_1\colon X\to Y$ is projection to the first axis, then $\pi_1(X)$ is conull and analytic. Thus by Theorem 4.1.33 we can find $\phi\colon Y\to X$ universally measurable such that $\pi_1\circ\phi=\operatorname{Id}$. Let $U\colon Y\to\mathcal{U}(H,K)$ be such that $\phi(y)=(y,U(y))$. By construction

$$U(y)^*N_yU(y)\subseteq M_y$$
$$U(y)^*M_yU(y)\subseteq N_y$$

so $U(y)$ has the desired properties. □

Theorem 3.1.10. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let $(M_y,H_y),(N_y,K_y)$ be two measurable fields of von Neumann algebras over Y, set*

$$M=\int_Y^{\oplus} M_y\,d\mu(y), N=\int_Y^{\oplus} N_y\,d\mu(y).$$

Suppose $M_y\cong N_y$ for almost every y. Then there are normal $*$-isomorphism $\pi_y\colon M_y\to N_y$, defined for almost every y, such that $y\to\pi_y(a(y))\in N$ for all $a\in M$. Thus $\pi=\int_Y^{\oplus}\pi_y\,d\mu(y)$ is an isomorphism between M and N.

Proof. As in the proceeding proof we may assume that $H_y=H, K_y=K$ are fixed separable Hilbert spaces and that $y\to M_y, N_y$ are Borel. By the essential uniqueness of a representation of a von Neumann algebra, we have that for almost every y, there is a a projection $p_y\in M_y'\otimes B(l^2(\mathbb{N}))$ and a unitary $U_y\colon p_y(H\otimes l^2(\mathbb{N}))\to K$, such that $U_y(p_y(M_y\overline{\otimes}\mathbb{C})p_y)U_y^*=N_y,(p_y(M_y\overline{\otimes}\mathbb{C})p_y)=U_y^*N_yU_y$. We will apply the measurable selection theorem as in the last theorem, but the one trick is to choose p_y measurably so that

$$M_y\to p_y(M_y\otimes\mathbb{C})p_y$$

is an isomorphism for almost every y. However $M_y\to p_y(M_y\otimes\mathbb{C})p_y$ is an isomorphism for almost every y, if and only if its adjoint is isometric on $(p_y(M_y\otimes\mathbb{C})p_y)$. Thus we first show that we can choose a sequence $\phi_n\colon Y\to B(H)_*$ which are Borel and such that $\phi_n(y)\big|_{p_y(M_y\otimes\mathbb{C})p_y}$ are norm dense in $(p_y(M_y\otimes\mathbb{C})p_y)_*$ for almost every y.

Let $\phi_n\in B(H\otimes l^2(\mathbb{N}))_*$ be a norm dense sequence.

Let $a_n(y), b_n(y)$ be Borel operator fields which are weak* dense in the unit ball of M_y, N_y for every y. Set X to be the set of $(y,v,p)\in Y\times B(H\otimes l^2(\mathbb{N}),K)\times B(H\otimes l^2(\mathbb{N}))$ such that

(a) p is a projection in $M_y'\otimes B(l^2(\mathbb{N})$
(b) $v^*v=p, vv^*=1$
(c) $v^*b_n(y)v\in p(M_y\overline{\otimes}\mathbb{C})p$

(d) $vp(a_n(y) \otimes 1)v^* \in N_y$

(e) $\|p\phi_n(y)p\big|_{p(M_y\overline{\otimes}\mathbb{C})p}\| = \|\phi_n(y)\big|_{M_y\overline{\otimes}\mathbb{C}}\|$

(f) $(\phi_n(y) \otimes 1)\big|_{M_y\otimes\mathbb{C}} = p\psi p$ for some $\psi \in (p(M \otimes \mathbb{C})p)_*$. where $p\psi p$ is defined by

$$(p\psi p)(x) = \psi(pxp).$$

Let X_0 be the set of all (y, p, v) satisfying $(a) - (e)$ Since

$$\|\phi_n(y)\big|_{p(M_y\overline{\otimes}\mathbb{C})p}\| = \sup_n \|\phi_n(y)(p(a_n(y) \otimes 1)p)\|$$

we see as in the previous theorem that X is Borel. Since

$$X = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{j=1}^{\infty} \{(y, v, p) : \|(p\phi_j(y)p)\big|_{M_y\otimes\mathbb{C}} - (\phi_n(y))\big|_{M_y\otimes\mathbb{C}}\| < 1/m\},[7]$$

the same logic as above shows that X is Borel. As in the previous theorem, if $\pi_1 \colon X \to Y$ is the projection onto the first axis, then $\pi_1(X)$ is analytic and conull, and we can use the measurable selection theorem to find

$$p \colon Y \to B(H \otimes l^2(\mathbb{N}))v \colon Y \to B(H \otimes l^2(\mathbb{N}), K)$$

such that

$$p(y) \in M_y' \otimes B(l^2(\mathbb{N}))\text{is a projection,}$$
$$v(y)^* v(y) = p, v(y)v(y)^* = 1$$
$$v^*(y) N_y v(y) \subseteq p(M_y\overline{\otimes}\mathbb{C})p$$
$$v(y)(p(y)(M_y\overline{\otimes}\mathbb{C})p(y))v(y)^* \subseteq N_y$$
$$\|p(y)\phi p(y)\big|_{p(y)(M_y\otimes\mathbb{C})p(y)}\| = \|\phi\big|_{M_y\otimes\mathbb{C}}\|.$$

Thus

$$\pi_y(x) = v(y)(p(y)(x \otimes 1)p(y))v(y)^*$$

has the desired properties.

□

Theorem 3.1.11. *Let A be a separable C^*-algebra, and let $\pi \colon A \to B(H)$ be a non-degenerate representation, with H separable. Let $B \subseteq \pi(A)'$ be any abelian subalgebra and write $B \cong L^\infty(Y, \mu)$ with Y compact Hausdorff and μ a finite Borel measure on Y. Then, there is a measurable field over Y of representations π_y of A, on H_y and a unitary $U \colon H \to \int_Y^\oplus H_y\, d\mu(y)$ such that*

$$U\pi U^* = \int_Y^\oplus \pi_y\, d\mu(y).$$

$$UAU^* = \int_Y^\oplus \mathbb{C}\, d\mu(y).$$

Further B is a maximal abelian subalgebra of $\pi(A)'$ if and only if π_y is irreducible for almost every y. In particular, every nondegenerate representation of A is a direct integral of irreducible representations. Thus, every unitary representation of a locally compact group is a direct integral of irreducible representations.

---

[7]Here it is crucial that (e) guarantees that $\phi \to p\phi p$ is isometric. Note that this extra condition just says that the image of the map $(p(M_y \otimes \mathbb{C})p)_* \to (M_y)_*$ is dense.

Proof. The existence of the disintegration follows form Theorems 2.1.10 and 2.2.4. Thus we focus on equivalence of B being maximal abelian and π_y being irreducible for almost every y.

First let us establish that $\{y : \pi_y \text{ is irreducible}\}$ is measurable. By applying Theorem 2.1.9, we see that we may assume that there is a partition $Y_n, n \in \mathbb{N} \cup \{\infty\}$ such that $H_y = \mathbb{C}^n$ ($\mathbb{C}^\infty = l^2(\mathbb{N})$) for $y \in \mathbb{C}^n$. Modifying π_y on a measure zero set we may assume that $y \to \pi_y(a)$ is strongly* Borel for $y \in Y_n$, and for all n (see Lemma 2.2.8). Let a_k be a dense sequence in A, and ξ_n a dense sequence in H, then

$$X_n = \{(y,T) \in Y_n \times B(H) : \|T\| = 1, T \in \pi_y(A)', T \notin \mathbb{C}1\} =$$

$$\bigcap_{k=1}^{\infty} \{(y,T) \in Y_n \times B(H) : \|T\| = 1, [a_n(y), T] = 0\} \cap Y_n \times [(B(H) \cap \overline{B(0,1)}) \setminus \mathbb{C}1].$$

is a Borel set. If $\pi_1 \colon X_n \to Y, \pi_2 \colon X_n \to B(H)$ are the projections we see that

$$\pi_1(X_n) = \{y \in Y_n : \pi_y \text{ is not irreducible }\}$$

and this set is analytic, so our claim follows. Throughout the rest of the proof we will assume that Y_n is partitioned as above and we will use the notation we just established.

Suppose that there is a positive measure set of y on which π_y is not irreducible. Then for some n, we have that $\mu(\pi_1(X_n)) > 0$, fix such an n. By Theorem 4.1.33 we may find a $\phi \colon \pi_1(X_n) \to X_n$, universally measurable and such that $\pi_1 \circ \phi = \operatorname{Id}$. Let $T(y) = \chi_{\pi_1(X_n)} \pi_2 \circ \phi$, then $T(y)$ is a decomposable operator in $\pi(A)'$, thus $T(y)$ commutes with

$$\int_Y^{\oplus} \mathbb{C}\, d\mu(y) = B$$

but is not in B, by construction. Thus B is not maximal abelian.

Conversely, suppose that B is not maximal abelian, and let $T \in B' \setminus B$. Since B is the diagonal algebra relative to the decomposition $\int_Y^{\oplus} H_y \, d\mu(y)$, Theorem 2.2.4 implies that we can write

$$T = \int_Y T_y \, d\mu(y).$$

We claim that $\{y : T_y \notin \mathbb{C}1\}$ is measurable. As above we may assume that $y \to T_y$ is Borel for $y \in Y_n$. Then

$$\{y : T_y \in \mathbb{C}1\}$$

is the projection to the first axis of the Borel set

$$\bigcup_{n=1}^{\infty} \{(y, \lambda) \in Y_n \times \lambda : T_y = \lambda\},$$

and is thus analytic, in particular measurable. Since $T \notin \int_Y^{\oplus} T_y \, d\mu(y)$ we must have that

$$\mu(\{y : T_y \notin \mathbb{C}1\}) > 0.$$

But then for almost every y in $\{y : T_y \notin \mathbb{C}1\}$ we have that $T_y \in \pi_y(A)'$. Thus we have found a positive measure set on which $\pi_y(A)'$ is not irreducible.

□

Theorem 3.1.12 (The Factor Decomposition)**.** *Every von Neumann algebra with separable predual is a direct integral of factors.*

Proof. Let M be a von Neumann algebra with separable predual, and represent M faithfully on a separable Hilbert space H. Let A be a separable unital weak* dense C^*-subalgebra of M (with the same unit) and let a_n be a dense sequence in A. [8] By the preceding corollary we can write $H = \int_Y^{\oplus} H_y \, d\mu(y)$ in such a way that

$$Z(M) = \int_Y^{\oplus} \mathbb{C} \, d\mu(y)$$

and the identity representation of A decomposes

$$\mathrm{Id} = \int_Y^{\oplus} \pi_y \, d\mu(y).$$

Set $M_y = \pi_y(A)''$. Then

$$M = A'' \subseteq \int_Y^{\oplus} \pi_y(A)'' \, d\mu(y),$$

by Theorem 3.1.5. However if $T = \int_Y^{\oplus} T_y \, d\mu(y) \in \int_Y^{\oplus} \pi_y(A)'' \, d\mu(y)$ and $S \in A'$, then since $A'' \supseteq Z(M)$, we see that $S \in Z(M)'$. Since $Z(M)$ is the diagonal algebra, we can write $S = \int_Y^{\oplus} S_y \, d\mu(y)$, by Theorem 2.2.4. Since $S \in \pi(A)'$, we have $[S, a_n] = 0$ for all n, so $[S(y), a_n(y)] = 0$ for almost every y. Thus $S_y \in \pi_y(A)'$ almost everywhere and thus $[T_y, S_y] = 0$ almost everywhere. So $[T, S] = 0$. Thus $T \in A''$, therefore

$$M = A'' = \int_Y^{\oplus} \pi_y(A)'' \, d\mu(y).$$

Thus, by Theorem 3.1.5

$$\int_Y^{\oplus} \mathbb{C} \, d\mu(y) = Z(M) = M \cap M' = \int_Y^{\oplus} M_y \cap M_y' \, d\mu(y),$$

so $M_y \cap M_y' = \mathbb{C}$ almost everywhere. That is, M_y is a factor for almost every y. □

3.2. **Direct Integrals and Type Classification.** In this section we show that a direct integral $\int_Y^{\oplus} M_y \, d\mu(y)$ is type I (resp. II,resp. III) if and only if almost every M_y is I (resp.II,resp.III). We similarly show that $\int_Y M_y \, d\mu(y)$ is finite if and only if M_y is finite almost everywhere, which also establishes the analogous claims for types II_1, II_∞. To do this we shall use the measurable selection Theorem (Theorem 4.1.33) about (1000)! times. We first start with a few simple results.

Proposition 3.2.1. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M = \int_Y^{\oplus} M_y \, d\mu(y)$.*

(i) Two projections p, q are equivalent if and only if, for almost every y we have $p(y)$ is equivalent to $q(y)$.

(ii) M is finite if and only if M_y is finite for almost every y.

[8] For existence of such an A, note that since M_* is separable we can find $a_n \in M$ such that for all $\phi \in M_*$ we have $\phi(a_n) \neq 0$ for some n. Then by the Hahn-Banach theorem, the linear span of a_n is weak* dense. Now let $A = C^*(\{a_n : n \in \mathbb{N}\}, 1)$.

Proof. (i) Suppose $v \in M$ and $v^*v = p, vv^* = q, v \in M$ it the follows that $v(y)^*v(y) = p(y), v(y)v(y)^* = q(y)$ for almost every y. Thus $p(y)$ is equivalent to $q(y)$ for almost every y. For the converse, by applying Theorem 2.1.9 and working on each direct summand, we may assume that M_y is represented on a fixed separable Hilbert space H for each y. Modifying p, q on null sets we may assume that $y \to p(y), q(y)$ are strongly* Borel and that $p(y), q(y)$ are projections for every y. Let

$$X = \{(y, v) \in Y \times B(H) : v \in M_y, v^*v = p(y), vv^* = q(y)\}$$

since $v^*v = p(y), vv^* = q(y)$ implies that $\|v\| \leq 1$, and $x \to x^*x$ is a Borel map on $\{x : \|x\| \leq 1\}$ we see by Proposition 3.1.2 that X is a Borel set. If π_1 is the projection onto the first coordinate we have that $\pi_1(X)$ is analytic and conull by assumption. By the measurable selection theorem we can find universally measurable map $v \colon \pi_1(X) \to \{T \in B(H) : \|T\| \leq 1\}$ such that $(y, v(y)) \in X$ for all y. Then $v \in M$ and implements the equivalence between $p(y)$ and $q(y)$ for almost every y.

(ii) If M_y is finite almost everywhere, it is easy to see from (i) that M is finite. Conversely, suppose that M is finite, then since Y is second countable we have that $\int_Y H_y \, d\mu(y)$ is separable we have that M_* is separable. Because M_* is separable and M is finite, we can find a faithful finite normal trace

$$\tau \colon M \to \mathbb{C}.$$

By Theorem 3.1.5 we can write $\tau = \int_Y^\oplus \tau_y \, d\mu(y)$. Let $a_n(y)$ be a weak* dense sequence of measurable fields of operators in M. If $E \subseteq Y$ is measurable we have that

$$\int_E \tau_y(a_n(y)a_m(y)) \, d\mu(y) = \tau(\chi_E a_n a_m) = \tau(a_m a_n \chi_E) = \int_E \tau_y(a_m(y)a_n(y)) \, d\mu(y)$$

thus $\tau_y(a_n(y)a_m(y)) = \tau_y(a_m(y)a_n(y))$ for almost every y. Taking a countable intersection of conull sets we see that we may assume that $\tau_y(a_n(y)a_m(y)) = \tau_y(a_m(y)a_n(y))$ for every y and every n, m. By the weak* density of $a_n(y), a_m(y)$ and the normality of τ_y we see that τ_y is a trace for every y. It is also faithful for almost every y, by Theorem 3.1.6. Thus for almost every y, we have that τ_y is a faithful finite normal trace, and thus M_y is finite.

□

Corollary 3.2.2. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M = \int_Y^\oplus M_y \, d\mu(y)$. Then M is properly infinite if and only if M_y is properly infinite for almost every y.*

Proof. Suppose M_y is properly infinite for almost every y. Let $z \in M$ be a central projection such that zM is finite. Then by the above proposition we have that $z(y)$ is finite for almost every y. Since M_y is properly infinite for almost every y, this implies that $z(y) = 0$ for almost every y, i.e. that $z = 0$ and M is properly infinite.

Conversely, suppose that M is properly infinite. Working on direct summands, we may assume that each M_y is represented on a fixed separable Hilbert space H and that $y \to M_y$ is Borel. Note that M_y is not properly infinite if and only if there exists $z \in Z(M_y)$ and $\tau \colon zM_y \to \mathbb{C}$ a nonzero trace. Indeed if τ, z as above exists, then $\{x \in M_y : \tau(x^*x) = 0\}$ is a weak* closed two-sided ideal in zM_y, so equals $z_o M_y$ for $z_0 \leq z, z_0 \in Z(M)$. Thus $1 - z_0$ is a nonzero central finite projection, and thus M_y is not properly infinite. Thus let X be the set of all $(y,z,\phi)\in Y\times B(H)\times B(H)_*$ such that

$$z\in Z(M_y), z \text{ a projection}$$

$$\phi\big|_{zM_y} \text{ is a tracial state.}$$

We claim that X is Borel, since the set of projections in $B(H)$ is a Borel subset of $\{T\in B(H): \|T\|\leq 1\}$, and the proof in Proposition 3.1.2 establishes that

$$\{(y,q,\phi):\ \text{q a projection}\phi\big|_{qM_yq} \text{ is a trace }\}$$

is Borel, we only need to show that $\{(y,z): z\in Z(M_y)\}$ is Borel, and that the set of all

$$\{(y,q,\phi): q \text{ a projection } \phi\big|_{qM_yq} \text{ is a state}$$

are Borel. If $b_n(y)$ are strongly* Borel, closed under multiplication, addition, adjoints and scaling by elements in $\mathbb{Q}[i]$, are such that $M_y=\overline{\{b_n(y):n\in\mathbb{N}\}}^{SOT}$, then the two sets in question are

$$\{(y,z): z\in Z(M_y)\}=\bigcap_{n=1}^{\infty}\{(y,z): zb_n(y)=b_n(y)z\}$$

$$\bigcap_{n=1}^{\infty}\{(y,q,\phi): q \text{ is a projection }, \phi(qb_n(y)^*qb_n(y)q)\geq 0, \phi(q)=1\}$$

and are thus Borel. So X is Borel, if $\pi_1\colon X\to Y$ is the projection onto the first axis, then

$$\pi_1(X)=\{y: M_y \text{ is not properly infinite}\}$$

is thus analytic.

Thus by the measurable selection theorem, we can find universally measurable $z\colon Y\to B(H), \tau\colon Y\to B(H)_*$ such that $(y,z(y),\tau(y))\in X$ for all $y\in\pi_1(X)$. If we let $f\in L^1(Y,\mu)$ with $f(y)>0$ for all y, then τf gives a faithful trace on zM. Thus $z=0$, i.e. $z(y)=0$ almost everywhere so $\pi_1(X)$ is null and this completes the proof.

□

Theorem 3.2.3. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M=\int_Y^{\oplus} M_y\,d\mu(y)$. Then M is semifinite if and only if M_y is semifinite for almost every y.*

Proof. Suppose that M is semifinite, since M is semifinite and has seperable predual, we can find $p_n\in M$ such that $p_n\nearrow 1$, and p_nMp_n is finite. Then $p_n(y)$ is increasing for almost every y, and it easy to see that for almost every y,

$$1=(\sup p_n)(y)=\sup(p_n(y))$$

so that $p_n(y)\nearrow 1$ for almost every y. By the above proposition we have that $p_n(y)M_yp_n(y)$ is finite for almost every y. Since $p_n(y)\nearrow 1$ for almost every y, this implies that M_y is finite for almost every y.

Conversely, suppose that M_y is semifinite for almost every y. Applying Theorem 2.1.9 we see that we may assume that M_y is each represented on a fixed separable Hilbert space H, and we may also assume that $y\to M_y$ is Borel. Note that a von Neumann algebra N is semifinite if and only if for every projection $p\in N$, we have that there is a projection $q \leq p$ and a nonzero normal trace $\tau\colon qNq \to \mathbb{C}$. Indeed if $J = \{x \in qNq : \tau(x^*x) = 0\}$, then J is weak* closed two-sided ideal in qNq so is of the form zN for some $z \in Z(qNq)$, and $1 - z$ is a finite projection under q, and therefore under p. So fix $p \in M$, and represent p by a strongly* Borel map $y \to p(y)$. Let X be the set of all $(y, q, \phi) \in Y \times B(H) \times B(H)_*$ such that

(a) $q \in M_y, q \leq p(y),\ q$ is a projection

(b) $\phi\big|_{qM_yq}$ is a positive trace, $\phi(q) = 1$, we claim that X is Borel. Since the set of projections in $B(H)$ is a Borel subset of $\{T \in B(H) : \|T\| \leq 1\}$, and by Proposition 3.1.2 $\{(y, q) : q \in M_y\}$ is Borel, and the proof in Proposition 3.1.2 establishes that $\{(y, q, \phi) : \phi\big|_{qM_yq}$ is a trace $\}$ is Borel, it remains to establish that $\{(y, q) : q \leq p(y)\}$ is Borel. But $\{(y, q) : q \leq p(y)\}$ is the inverse image of $B(H)_+$ under the Borel map $(y, q) \to p(y) - q$, so is Borel. Thus X is Borel. If $\pi_1\colon X \to Y$ is the projection onto the first coordinate, then $\pi_1(X) \supseteq \{y : M_y \text{ is semifinite}\}$ is conull and analytic, and the measurable cross section theorem says we can find universally measurable maps

$$q\colon Y \to B(H) \setminus \{0\}, \tau\colon Y \to B(H)_*$$

such that $(y, q(y), \tau(y)) \in B(H)_*$ for every $y \in \pi_1(X)$. Thus $q \leq p$ is a projection in M, and if $f \in L^1(Y, \mu)$ with $f(y) > 0$ for every y, then $f\tau$ is a nonzero trace on qMq. Therefore our earlier remark implies that M is semifinite.

□

Corollary 3.2.4. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M = \int_Y^{\oplus} M_y\, d\mu(y)$. Then M is type III if and only if M_y is type III for almost every y.*

Proof. If M is not type III, then we can $z \in Z(M)\setminus\{0\}$ such that zM is semifinite. As

$$zM = \int_Y^{\oplus} z(y) M_y\, d\mu(y)$$

the previous theorem implies that $z(y)M_y$ is semifinite for almost every y. Since $z \neq 0$, we have that $z(y) \neq 0$ for a positive measure set of y, but then we can find some y such that M_y is type III and $z(y) \neq 0$, which is a contradiction.

Conversely, suppose M is type III. As before, we may assume that M_y is represented on a fixed separable Hilbert space H and that $y \to M_y$ is Borel. Suppose that it is not the case that M_y is type III for almost every y. As in the above theorem, let X be the set of all $(y, p, \phi) \in Y \times B(H) \times B(H)_*$ such that p is a projection in M_y, and $\phi\big|_{pM_yp}$ is a positive trace and $\phi(p) = 1$. As in the last theorem $\pi_1(X) = \{y : M_y \text{ is not type } III\}$ is analytic, and we can find $p\colon \pi_1(X) \to B(H) \setminus \{0\}, \tau\colon \pi_1(X) \to B(H)_*$ such that $(y, p(y), \tau(y)) \in X$. By assumption $\mu(\pi_1(X)) > 0$, and arguing as in the above theorem we see that we can find a nonzero trace on pMp. This contradicts the assumption that M is type III.

□

Theorem 3.2.5. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M = \int_Y^{\oplus} M_y\, d\mu(y)$. Then M is type II if and only if M_y is type II for almost every y.*

Proof. Suppose first that M_y is type II for almost every y. Let $p \in M$ be an abelian projection. Then we have that

$$\int_Y^{\oplus} p(y)M_y p(y)\,d\mu(y) = pMp \subseteq (pMp)' = pM'p = \int_Y^{\oplus} p(y)M'p(y)\,d\mu(y).$$

Thus by Theorem 3.1.5 we have that $p(y)M_y p(y) \subseteq p(y)M'p(y)$ almost everywhere, i.e. $p(y)$ is abelian for almost every y. Since M_y is type II almost everywhere, this implies that $p(y) = 0$ for almost every y.

Conversely, suppose that M is type II. As usual by working on direct summands we may assume that M_y is represented on a fixed separable Hilbert space H and that $y \to M_y$ is Borel. Let

$$X = \{(y,p) \in Y \times B(H) : p \in M_y,\ p \text{ is a projection and } pM_yp \text{ is abelian}\}$$

we claim that X is a Borel. Since we have already seen (see Proposition 3.1.2) that

$$\{(y,p) \in Y \times (B(H) \setminus \{0\}) : p \in M_y,\ p \text{ is projection }\}$$

it remains to show that

$$\{(y,p) : pM_yp \text{ is abelian }\}$$

is Borel. But if $a_n(y)$ are strongly* Borel maps such that $M_y = \overline{\text{Span}\{a_n(y) : n \in \mathbb{N}\}}^{SOT}$ for every y, then

$$\{(y,p) : pM_yp \text{ is abelian }\} = \bigcap_{n,m} \{(y,p) : pa_n(y)pa_m(y)p = pa_m(y)pa_n(y)p\}$$

and is Borel, since $(y,p) \to p, (y,p) \to a_n(y)$ are Borel. If π_1 is projection to the first coordinate, then

$$\pi_1(X) = \{y : M_y \text{ is not type } II\}$$

is analytic, and the measurable selection theorem implies that we can find $p\colon Y \to B(H) \setminus \{0\}$ universally measurable such that $(y, p(y)) \in X$ for every y. Thus $\chi_{\pi_1(X)}p \in M$ is an abelian projection, and we must have that $p = 0$, i.e. that $p(y) = 0$ almost everywhere. Since $p(y) \neq 0$ for all $y \in \pi_1(X)$ this is only possibly when $\mu(\pi_1(X)) = 0$. Therefore M_y is type II almost everywhere.

□

Theorem 3.2.6. *Let M be a von Neumann algebra with seperable predual and by Theorem 3.1.12 write*

$$M = \int_Y M_y\,d\mu(y)$$

with M_y factors, and Y a second countable locally compact Hausdorff space and μ a finite measure on Y which is nonzero on all nonempty open sets. Then

$$E_I = \{y : M_y \text{ is type } I\}$$
$$E_{II} = \{y : M_y \text{ is type } II\}$$
$$E_{III} = \{y : M_y \text{ is type } III\}$$

are measurable sets and

$$z_I = \chi_{E_I}$$
$$z_{II} = \chi E_{II}$$
$$z_{III} = \chi_{E_{III}}$$

are such that

$$z_I M \text{ is type } I$$
$$z_{II} M \text{ is type } II$$
$$z_{III} M \text{ is type } III.$$

A similar remark applies to $E_{II_1} = \{y : M_y \text{ is type } II_1\}, E_{I_\infty} = \{y : M_y \text{ is type } II_\infty\}$. *In particular, every von Neumann algebra with separable predual of type I (resp.* I_∞, II_1, II_∞, III*), is a direct integral of factors of type I,(resp.* I_∞, II_1, II_∞, III*).*

Proof. We know that we can find some z_I, z_{II}, z_{III} central projections such that

$$1 = z_I + z_{II} + z_{III}$$

and

$$z_I M, z_{II} M, z_{III} M \text{ are of types } I, II, III \text{ respectively}.$$

Since z_I, z_{II}, z_{III} are central they must be in $\int_Y^{\oplus} \mathbb{C}\, d\mu(y)$, and so must be characteristic functions of sets, say F_I, F_{II}, F_{III}. All we have to argue is that $z_{II} = \chi_{E_{II}}, z_{III} = \chi_{E_{III}}$ almost everywhere. By Corollary 3.2.4 and Theorem 3.2.5 we have that for almost every $y \in F_{III}$ (resp. F_{II}) we have that M_y is type III (resp. II). Moreover by Theorem 3.2.3 we have that for almost every $y \in F_{II} \cup F_I$ that M_y is semifinite, i.e. not of type III, so $\mu(E_{III} \cap (F_I \cup F_{II})) = 0$ and $\mu(F_{III} \setminus (E_{III}) = 0$. Thus $F_{III} = E_{III}$ almost everywhere. Similarly, by Theorem 3.2.5 we have that $\mu(F_I \cup F_{III} \cap E_{II}) = 0$, and we already saw that $\mu(F_{II} \setminus E_{II}) = 0$. Thus $F_{II} = E_{II}$ almost everywhere, this completes the proof. □

Corollary 3.2.7. *Let M be a type I von Neumann algebra with separable predual, then*

$$M \cong L^\infty(X_0, \mu_0) \overline{\otimes} B(l^2(\mathbb{N})) \oplus_{n=1}^{\infty} L^\infty(X_n, \mu_n) \overline{\otimes} M_n(\mathbb{C}),$$

with μ_0 *a Radon probability measure on* X_0 *which is positive on all nonempty open sets.*

Proof. Write

$$M = \int_Y^{\oplus} M_y \, d\mu(y)$$

with M_y factors and the usual assumptions on Y. Then $E = \{y : M_y \text{ is isomorphic to } B(l^2(\mathbb{N}))\}$ is measurable by the above proposition and

$$M \cong \int_E^{\oplus} M_y \, d\mu(y) \oplus \int_{E^c}^{\oplus} M_y \, d\mu(y) \cong$$

$$L^\infty(E, \mu) \overline{\otimes} B(l^2(\mathbb{N})) \oplus \int_{E^c}^{\oplus} M_y \, d\mu(y).$$

Being an abelian von Neumann algebra with separable predual, we may write $L^\infty(E, \mu) \cong L^\infty(X_0, \mu_0)$ with X_0, μ_0 as in the statement of the proposition. By similar logic, it suffices to show that

$$\{y : \dim M_y = n^2\}$$

is measurable for $n \in \mathbb{N}$. First note that if τ_y is the unique tracial state on $N = M_y$ for $y \in E^c$, then $y \to \tau_y(a(y))$ is measurable for all $a \in \int_{E^c}^{\oplus} M_y \, d\mu(y)$. Indeed, since N is finite by Proposition 3.2.1, it has a faithful normal tracial state. By

Theorem 3.1.6, we may find a measurable field of faithful tracial states ϕ_y such that $\tau = \int_Y^\oplus \phi_y \, d\mu(y)$, in particular $\phi_y(1) > 0$ almost everywhere. By uniqueness,

$$\tau_y = \frac{\phi_y}{\phi_y(1)}$$

and this establishes the measurability of τ_y. Using the elements of M (and regarding $M_y = L^2(M_y, \tau_y)$ by finite dimensionality) as sections, we have that $L^2(M_y, \tau_y)$ is a measurable field of Hilbert spaces over E^c. By what we know about measurable fields of Hilbert spaces,

$$\dim L^2(M_y, \tau_y) = \dim M_y$$

is a measurable function. □

Theorem 3.2.8. *Let μ be a Radon measure on a second countable locally compact Hausdorff space Y, and let M_y be a measurable field of von Neumann algebras over Y, and let $M = \int_Y^\oplus M_y \, d\mu(y)$. Then M is type I if and only if M_y is type I for almost every y.*

Proof. Suppose that M_y is type II_1 for almost every y. Working on direct summands as before, we may assume that M_y is represented on a fixed separable Hilbert space H for each y, and that $y \to M_y$ is Borel. Let $z' \in M$ be a central projection, and choose a measure zero modification $z(y)$ such that $y \to z(y)$ is Borel and $z(y)$ is a projection for every y. If $b_n(y)$ are Borel fields which generate M_y for every y, and such that $b_n(y) = b_n(y)^*$, then

$$\{y : z(y) \in Z(M_y)\} = \bigcap_{n=1}^{\infty} \{y : z(y) b_n(y) = b_n(y) z(y)\}$$

and is thus Borel. So we may insist that $z(y) \in Z(M_y)$ for every y.

$X = \{(y, p) \in Y \times B(H) \backslash \{0\} : p \text{ is a projection}, p \leq z(y), p \in M_y, pM_y p \text{ is abelian } \}$

as in the previous theorem, we have that X is Borel. If π_1 is the projection onto the first axis, then $\pi_1(X) \supseteq \{y : M_y \text{ is type } I\}$ is conull and analytic. The measurable selection theorem implies we can find $p \colon \pi_1(X) \to M$ such that $(y, p(y)) \in X$ for all y. Then $p \in M$ is a nonzero abelian projection under z, and thus M is type I.

Conversely, suppose M is type I. Because M is type I and with separable predual, we may assume by the previous corollary that

$$M = L^\infty(X_0, \mu_0) \overline{\otimes} M_n(\mathbb{C}) \bigoplus_{n=1}^{\infty} L^\infty(X_n, \mu_n) \overline{\otimes} M_n(\mathbb{C})$$

with X_n a compact metric space and μ_n a Borel probability measure on X_n which is positive on all nonempty open sets. If we let z_n be such that $z_n M = L^\infty(X_n, \mu_n) \overline{\otimes} M_n(\mathbb{C})$, (with $M_\infty(\mathbb{C}) = B(l^2(\mathbb{N}))$), then it suffices to show that $z_n M_y$ is type I for almost every y. Thus we may assume that

$$M = L^\infty(X, \nu) \overline{\otimes} M_n(\mathbb{C})$$

where X is a compact metric space and ν is a Borel probability measure on X which is positive on nonempty open sets. Write

$$1 \otimes e_i = \int_Y^\oplus p_i(y) \, d\mu(y).$$

Then Proposition 3.2.1 implies that for almost every y, $p_i(y) \sim p_j(y)$ for all i, j. Also, as in the last Theorem we have that $p_i(y)$ is abelian for almost every y. Thus for almost every y we can write

$$1 = \sum_i p_i(y), p_i(y) \sim p_j(y) \text{ and } p_i(y) \text{ is abelian.}$$

This last equation implies that

$$M_y \cong p_1 M p_1 \overline{\otimes} M_n(\mathbb{C})$$

and since $p_1 M p_1$ is abelian for almost every y, this last equation implies that M_y is type I for almost every y.

□

4. Appendix: Polish Spaces and Measurable Selection

In the study of Direct Integrals, we frequently have to appeal to "measurable selection" theorems. A typical example is that of direct integrals of representations, if π_y, ρ_y are measurable fields of representations of a C^*-algebra such that $\pi_y \cong \rho_y$ for almost every y, we would like to assert that $\int^{\oplus} \pi_y \cong \int^{\oplus} \rho_y$. This is indeed true (if the C^*-algebra is separable), but in order to prove one cannot arbitrarily choose a unitary equivalence for each point, one needs to know that we can measurable choose a unitarily equivalence at each point. Similarly if have a direct integral of Von Neumann algebra $M = \int_Y^{\oplus} M_y \, d\mu(y)$, and two projections $p, q \in M$ such that p_y is Murray-von Neumann equivalent to q_y for almost every y, we would like to assert that p is equivalent to q. Similar remarks apply if we know that almost every M_y has a faithful trace, or is type I, II, III, etc. In this section we develop the necessary machinery to prove such a measurable selection theorem. This theorem belongs more to the field of *descriptive set theory* than to operator algebras, so we will have to develop some theory about Polish spaces, standard Borel spaces, analytic sets and so on. This may seem somewhat separated from our goal of studying direct integrals of Hilbert spaces and von Neumann algebras, but it will get us the theorem that we need.

Definition 4.1.9. A topological space X is called a *Polish space* if it is separable and there is a metric d on X, which gives the topology of X and such that (X, d) is a complete metric space

Definition 4.1.10. A *measurable space* is a set X equipped with a σ-algebra $\mathcal{M}$. A measurable space is said to be a *standard Borel space*if there is a Polish topology on X such that $\mathcal{M}$ is the set of Borel subsets of X with respect to this topology

Definition 4.1.11. A measure space $(X, \mathcal{M}\, \mu)$ is called a *standard measure space* if there is null set $N \subseteq X$ and a σ-algebra $\mathcal{M}_0$ on X such that $(X \setminus N, \mathcal{M}_0\big|_{X \setminus N})$ is a standard Borel space and $\mathcal{M}_0 \subseteq \mathcal{M} \subseteq \overline{\mathcal{M}_0}$. Here $\overline{\mathcal{M}_0}$ is the completion of $\mathcal{M}_0$ with respect to μ.

One might wonder why we make this definition a Polish space, and not just declare a Polish space to be a separable complete metric space. There are two main reasons, one is that we do not want to think about properties that depend on the metric in question, but will only be concerend with properties that depend upon the topology in questions. Indeed, we are going to be mainly concerned with properties that only depend upon the *Borel sets* in question. So we will primarily work with

Borel set, Borel functions, etc. and not so much with continuous functions, open set, closed sets and so on. The second reason, is that the notion of a Polish space abstracts the notion of a complete metric space and this becomes important when talking about subspaces. For example, we will show that a subset of a Polish space is a Polish space if and only if it is a G_δ set, whereas a subset of a complete metric space is a complete metric space if and only if it is a closed set. For example, this implies that the irrationals are a Polish space, whereas no one would claim that the irrationals are a complete metric space.

We first prove a theorem which gives the most important examples of Polish spaces, one which can surject onto every Polish space, and one in which every Polish space embeds. Note that if (X_n, d_n) are metric spaces with then so is $X = \prod_{n=1}^{\infty} X_n$ with metric

$$d((x_n),(y_n)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d(x_n, y_n)}$$

it is a standard exercise to show that d gives the product topology and is complete/separable if each (X_n, d_n) is complete/separable. In particular, countable products of Polish spaces are Polish. For later use, we also note that we can give the disjoint union

$$\coprod_{n=1}^{\infty} X_n$$

the metric

$$d(x,y) = \begin{cases} \frac{d_n(x,y)}{1+d_n(x,y)} \text{ if } x, y \in X_n \\ 1 \text{otherwise} \end{cases}.$$

In this case $\coprod_{n=1}^{\infty} X_n$ is complete and separable if each (X_n, d_n) is. So disjoint unions of Polish spaces are Polish spaces.

Theorem 4.1.12. *Every Polish space has a homeomorphic embedding into* $[0,1]^{\mathbb{N}}$, *and is the image of a continuous map from* $\mathbb{N}^{\mathbb{N}}$.

Proof. Let X be a Polish space with compatible complete metric d. Let x_n be a dense sequence in X and define

$$\phi \colon X \to [0,1]^{\mathbb{N}}$$

by $\phi(x) = \left(\frac{d(x_n,x}{1+d(x_n,x)}\right)_{n=1}^{\infty}$, note that ϕ is continuous, since each of its coordinates functions is. We claim that ϕ is injective and a homeomorphism onto is image. If $x \neq y$ in X, then sinc $\{x_n\}_{n=1}^{\infty}$ is dense, we can find an n such that $d(x_n, x) \neq d(x_n, y)$. Since $x \to \frac{x}{1+x}$ is injective (if you don't believe me, take a derivative) we find have that $(\phi(x))_n \neq (\phi(y)))_n$, and ϕ is injective. To show that ϕ is a homeomorphism onto its image, suppose that $\phi(x^{(n)}) \to \phi(y)$, then $(\phi(x^{(n)}))_k \to (\phi(y))_k$ for each k, i.e.

$$\frac{d(x_k, x^{(n)})}{1 + d(x_k, x^{(n)})} \to_{n\to\infty} \frac{d(x_k, y)}{1 + d(x_k, y)}$$

for each k. Since $x \to x/1 + x$ is a homeomorphism on $[0, \infty)$ this implies that $d(x_k, x^{(n)}) \to_{n\to\infty} d(x_k, y)$ for all k. Fix $\varepsilon > 0$ and choose k such that $d(x_k, y) < \varepsilon$, then for all n large we have $d(x_k, x^{(n)}) < \varepsilon$ and thus $d(x^{(n)}, y) < 2\varepsilon$ by the triangle inequality. Thus $x^{(n)} \to y$ and we have that ϕ is a homeomorphism onto its image.

For the other part, we set up some notation. If $\sigma = (a_1, \dots, a_k)$ is a finite sequence of positive integers we let $|\sigma| = k$, and $\sigma l = (a_1, \dots, a_k, l)$. For each finite sequence σ of positive integers we will construct non-empty open sets U_σ with satisfy the following properties:

(i) $\operatorname{diam}(\overline{U_\sigma}) = \sup\{d(x,y) : x, y \in \overline{U_\sigma}\} \leq \frac{1}{|\sigma|}$,

(ii) $U_\sigma = \bigcup_{n=1}^\infty U_{\sigma n}$.

(iii)$U_\varnothing = X$, i.e. we requrie that $\bigcup_{n=1}^\infty U_n = X$.

Set $U_\varnothing = X$. Suppose we have constructed U_σ for $|\sigma| \leq k$ satsifying (i) and satisfying (ii) when $|\sigma| < k$. For each σ, and $x \in U_\sigma$ we can find a ball of radius $\varepsilon(x)$, which we may assume to have $\varepsilon(x) < \frac{1}{2|\sigma|}$ such that $B(x, \varepsilon(x)) \subseteq U_\sigma$. Since X is a separable metric space, and is thus second countable, we have that U_σ is second countable, so the cover $\{B(x, \varepsilon(x))\}$ of U_σ has a countable subcover. Thus we can find $(x_n)_{n=1}^\infty$ such that $B(x_n, \varepsilon(x_n))$ cover U_σ. Setting $U_{\sigma n} = B(x_n, \varepsilon(x_n))$ completes the inductive step.

If $\sigma = (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ we denote $\sigma\big|_n = (a_1, \dots, a_n)$. Since (X, d) is complete and $\operatorname{diam}(\overline{U_{\sigma|_n}}) \to 0$ we can define $f\colon \mathbb{N}^{\mathbb{N}} \to X$ by saying that $f(\sigma)$ is the unique point in

$$\bigcap_{n=1}^\infty \overline{U_{\sigma|_n}}.$$

If $x \in X$, then we can find $n_1 \in \mathbb{N}$ such that $x \in U_{n_1}$, then since $x \in U_{n_1}$ we can find n_2 such that $x \in U_{n_2}$. Continuing inductively we find $\sigma = (n_1, n_2, \dots)$ with $x \in U_{\sigma|_k}$ for all k and $f(\sigma) = x$, thus f is surjective. To see that f is continuous, let $\sigma \in \mathbb{N}^{\mathbb{N}}$, and $\varepsilon > 0$ be given, and choose n such that $1/n < \varepsilon$. Let $U = \{\sigma' \in \mathbb{N}^{\mathbb{N}} : \sigma\big|_n = \sigma\big|_n\}$, then U is open by definition of the product topology. If $\sigma' \in U$ then since $\operatorname{diam}(\overline{U_{\sigma|_n}}) \leq \frac{1}{n}$ and $f(\sigma), f(\sigma') \in \overline{U_{\sigma|_n}}$ we have that $d(f(\sigma), f(\sigma')) \leq 1/n < \varepsilon$. Thus f is continuous, and the proof is complete. □

Because of this theorem, we can often reduce question about Polish spaces to questions about $\mathbb{N}^{\mathbb{N}}$, or $[0,1]^{\mathbb{N}}$. We next characterize when a subset of a Polish space is Polish.

Theorem 4.1.13. *A subset of a Polish space is Polish if and only if it is a G_δ set.*

Proof. Let X be a Polish space with compatible complete metric d. Suppose $E = \bigcup_{n=1}^\infty U_n$ with U_n open in X, and $U_{n+1} \subseteq U_n$. Define $f_n : E \to (0, \infty)$ by $f_n(x) = \frac{1}{d(x, U_n^c)}$. Define $\phi\colon E \to X \times (0,\infty)^{\mathbb{N}}$ by $\phi(x) = (x, f_1(x), f_2(x), \dots)$. Note that $(0, \infty)$ is a Polish space, begin homeomorphic to $\mathbb{R}$. All we have to show is that ϕ is a homeomorphism onto its image and that $\phi(E)$ is closed. It will then follow that E is homeomorphic to a closed subset of a Polish space, and is thus a Polish space. It is clear that $\phi(E)$ is injective, and that ϕ is continuous since each of its coordinate functions is, as ϕ^{-1} on $\phi(E)$ is just projection onto the first coordinate restricted to $\phi(E)$ it follows that ϕ^{-1} is continuous on $\phi(E)$. Thus it remains to show that $\phi(E)$ is closed. Suppose $\phi(x^{(k)}) \to y = (y_1, t_2, t_3, \dots)$ then $x^{(k)} \to y_1$, and if we show that $y_1 \in E$, then it follows by continuity of f_n that $f_n(x^{(k)}) \to f_n(y_1)$ so

$y = \phi(y_1)$. So we have to show that $y_1 \in U_n$ for each n. Fix such an n, we have that

$$\frac{1}{d(x^{(k)}, U_n^c)} \to_{k\to\infty} t_n > 0$$

by assumption. As

$$d(x_k, U_n^c) \to d(y, U_n^c)$$

we have that $d(y, U_n^c) > 0$ so $y \in U_n$. This proves that E is a Polish space.

Conversely, suppose that $E \subseteq X$ is Polish, and let d_0 be a complete compatible metric on E. Let E_n be the set of $x \in \overline{E}$ such that there exists an open neighborhoud U of x with $\operatorname{diam}_{d_0}(U \cap E) \leq 1/n$, where $\operatorname{diam}_{d_0}$ means the diameter measured with respect to d_0. By definition E_n is a neighborhood of each of its points and is thus open, we have that $\bigcap_{n=1}^{\infty} E_n \supseteq Y$. Suppose $x \in \bigcap_{n=1}^{\infty} E_n$, choose an open set U_n such that U_n is a neighborhood of X and $\operatorname{diam}_{d_0}(U_n \cap E) \leq 1/n$. Replacing U_n with $\bigcap_{j=1}^{n} U_j$, we may assume that U_n is decreasing. Since E is dense in $\overline{E}$, we have that $E \cap U_n \neq \varnothing$ for all n. Thus $U_n \cap E$ are a decreasing sequence of sets in E whose d_0-diameter tends to 0, so by completeness of d_0, we have $\bigcap_{n=1}^{\infty} U_n \cap E = \{y\}$ with $y \in E$. But $x \in \bigcap_{n=1}^{\infty} U_n \cap E$, so $x = y \in E$. Thus $\bigcap_{n=1}^{\infty} E_n = \{x\}$. Since each E_n is open in $\overline{E}$ we can find V_n open so that $E_n = V_n \cap \overline{E}$, setting $W_m = \{x \in X : d(x, E) < 1/n\}$ we have that

$$E = \bigcap_{n=1}^{\infty} V_n \cap \overline{E} = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} V_n \cap W_m$$

and is thus a G_δ set. □

Corollary 4.1.14. *A topological space is Polish if and only if it is homeomorphic to a G_δ subset of* $[0,1]^{\mathbb{N}}$.

Proof. Combine Theorems 4.1.12 and 4.1.13. □

Definition 4.1.15. Let X be a separable metrizable topological space. A set $E \subseteq X$ is *analytic* if there exists a Polish space and a continuous $f : Y \to X$ such that $f(Y) = E$. A set is coanalytic if its complement is analytic.

By taking a completion, we may as well assume that X is Polish. So we will often assume that our analytic sets are subsets of Polish spaces. We shall later see that an analytic set is measurable with respect to any measure on X. It turns out that not every analytic set is Borel.

Theorem 4.1.16. *Let X be a separable space. If $(X_n)_{n\in\mathbb{N}}$ are a disjoint family of analytic sets in X, then there is a disjoint family $(B_n)_{n\in\mathbb{N}}$ of Borel subsets of X such that $X_n \subseteq B_n$.*

Proof. Note that if $(A_n)_{n\in\mathbb{N}}, (A'_n)_{n\in\mathbb{N}}$ are subsets of X such that for all n, m there is a Borel $B_{n,m}$ with $B_{n,m} \subseteq A_n$ and $B_{n,m} \cap A'_n = \varnothing$, then there is a Borel B such that $B \supseteq \bigcap_{n=1}^{\infty} A_n$ and $B \cap (\bigcup_{m=1}^{\infty} A'_m) = \varnothing$. Indeed, we can set $B = \bigcup_{n=1}^{\infty} \left(\bigcap_{m=1}^{\infty} B_{nm}\right)$.

We first handle the case when (X_n) consists of two sets X_1, X_2 which are disjoint. By definition of analytic and Theorem 4.1.12 we can find $f : \mathbb{N}^{\mathbb{N}} \to X_1, g : \mathbb{N}^{\mathbb{N}} \to X_2$ which are continuous and surjective. For a finite sequence $\sigma = (n_1, \ldots, n_k)$ of integers set

$$B_\sigma = \{\sigma' \in \mathbb{N}^{\mathbb{N}} : \sigma'\big|_k = \sigma\},$$

with the notation as in Theorem 4.1.12. Then, agian using the notation in Theorem 4.1.12, we have
$$B_\sigma = \bigcup_{n=1}^{\infty} B_{\sigma n}.$$
Set
$$A_\sigma = f(B_\sigma)$$
$$A'_\sigma = g(B_\sigma).$$
Assume the claim is false, so that there is no Borel $B \supseteq X_1$ such that $B \cap X_2 = \varnothing$. Then by the first paragraph we can find $n_1, m_1 \in \mathbb{N}$ so that there is no Borel $B \supseteq A_{n_1}$ such that $B \cap A'_{m_1} = \varnothing$. By the same logic we can find n_2, m_2 such that for all Borel $B \supseteq A_{n_1,n_2}$ we have $B \cap A'_{m_1 m_2} \neq \varnothing$. Inductively we construct $\sigma, \sigma' \in \mathbb{N}^{\mathbb{N}}$ such that for all k we have that if $B \supseteq A_{\sigma\big|_k}$ then $B \cap A'_{\sigma'\big|_k} \neq \varnothing$. Let $x = f(\sigma) \in X_1, x' = f(\sigma') \in X_2$. Since $x \neq x'$, we can find disjoint open sets $V, W \subseteq X$ such that $x \in V, x' \in W$. Thus $f^{-1}(V), f^{-1}(W)$ are open neighborhoods of σ, σ' respectively. This implies that we can find k such that
$$B_{\sigma\big|_k} \subseteq f^{-1}(V), B_{\sigma'\big|_k} \subseteq f^{-1}(W).$$
But then
$$A_{\sigma\big|_k} \subseteq V, A'_{\sigma'\big|_k} \subseteq W$$
so that $A_{\sigma\big|_k}, A'_{\sigma'\big|k}$ are seperated by V, which is contrary to our construction. Thus we can find a Borel B such that $X_1 \subseteq B$ and $X_2 \cap B' = \varnothing$.

By the above argument, for each n, m we can find a Borel B_{nm} so that $X_n \subseteq B_{nm}$ and $X_m \cap B_{nm} = \varnothing$. Define $B_1 = \bigcap_{m=2}^{\infty}$ and define B_n inducitvely by
$$B_n = \left(\bigcap_{m=n+1}^{\infty} B_{n,m} \right) \setminus \left(\bigcup_{j=1}^{n-1} B_j \right).$$
Then B_n is a disjoint sequence of Borel sets such that $X_n \subseteq B_n$. □

Corollary 4.1.17. *If X is a separable metric space, then an analytic set is Borel if its complement is analytic.*

Proof. Suppose A, A^c are analytic. Then by the above we can find disjoint Borel sets B_1, B_2 in X such that $A \subseteq B_1, A^c \subseteq B_2$. Since $X = A \cup A^c$ this implies that $B_1 = A, B_2 = A^c$, thus A is Borel. □

We would like to prove the converse to this corollary, but first we will need a few lemmas.

Lemma 4.1.18. *Let X be a Polish space. There is a Polish space P, with a countable basis of open and closed sets and a bijective continuous map $f\colon P \to X$.*[9]

Proof. We do this in the following steps.

(i) If $(X_n)_{n\in\mathbb{N}}$ are disjoint Polish subsets of X, and the Lemma is true for X_n, it is true for $\bigcup_{n=1}^{\infty} X_n$. [10]

[9]Note that we are *not* asserting that f is a homeomorphism.

[10]Here, of course $\bigcup_{n=1}^{\infty} X_n$ is not necesseraily Polish. So by "the lemma is true for $\bigcup_{n=1}^{\infty} X_n$," we simply mean that there is a Polish space, having a countable basis of open and closed sets which continuously bijects onto $\bigcup_{n=1}^{\infty} X_n$.

(ii) If the Lemma is true for Polish spaces $(X_n)_{n\in\mathbb{N}}$ then it is true for $\prod_{n=1}^{\infty} X_n$.

(iii) If X is Polish and the Lemma is true for $X_n \subseteq X$, then it is true for $\bigcap_{n=1}^{\infty} X_n$.

(iv) If X is Polish and the Lemma is true for X, then it is true for any open subset of X.

(v) The Lemma is true for $X = [0,1]$.

Note that once $(ii)-(v)$ have been established the lemma is proven by Theorem 4.1.12. To prove (i), let P_n be a Polish space with a countable basis of open and closed sets and $\phi_n\colon P_n \to X_n$, a continuous bijection. Set $P = \coprod_{n=1}^{\infty} P_n$ and define $\phi\colon P \to \bigcup_{n=1}^{\infty} X_n$ by $\phi\big|_{P_n} = \phi_n$, then P is a Polish space having the desired property and ϕ bijects continuously onto $\bigcup_{n=1}^{\infty} X_n$. The proof for (ii) is identical. To prove (iii) again let P_n be Polish spaces with a countable basis of open and closed sets and let $\phi_n\colon P_n \to P$ be a continuous bijection. Let $P' = \prod_{n=1}^{\infty} P_n$ and set

$$P = \{y \in P : f_1(y_1) = f_2(y_2) = \cdots\}$$

then P is a Polish space having the desired property and $\phi\colon P \to \bigcap_{n=1}^{\infty} X_n$ given by $\phi(y) = \phi_1(y_1)$ is a continuous bijection. For (iv), suppose P is a Polish space with a countable basis of open and closed sets, and let $\phi\colon P \to X$ be a continuous bijection. If $U \subseteq X$ is open, then $P' = \phi^{-1}(U) \subseteq P$ is open, hence Polish, and being open it too has a countable basis of open and closed sets. Further $\phi\big|_{P'}$ is a continuous bijection onto U.

Finally we establish (v). Let A be the set of irrationals in $[0,1]$. Then $A \cap (r,s)$ with $r < s$ rational, is a countable basis of open and closed sets in A. Further $[0,1]$ is the union of A and the $\mathbb{Q} \cap [0,1]$ which is a countable disjoint union of one point sets, so (i) completes the proof. □

We are now ready to prove the converse of Corollary 4.1.17.

Theorem 4.1.19. *Let X be a Polish space and B a Borel subset of X. Then there is a Polish space P, with a countable basis of open and closed sets and a continuous bijective map $f\colon P \to B$. In particular a Borel set is analytic (and coanalytic).*

Proof. Let $\mathcal{F}$ be the set of all sets $E \subset X$ for which there is a Polish space with a countable basis of open and closed sets and a continuous bijective map $f\colon P \to E$, and such that E^c has the same property. As in the proof of the above Lemma, we have that if $E_n \in \mathcal{F}$ then $\bigcap_{n=1}^{\infty} E_n$ is the image of a bijection from a Polish space with a countable basis of open and closed sets. If we set $B_n = E_n \cap \left(\bigcap_{k=1}^{n-1} A_k^c\right)$ then as in the preceeding lemma, we can find a bijection from a Polsih space onto B_n for each n, and since B_n is disjoint and

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$$

it follows that $\bigcup_{n=1}^{\infty} A_n$ has the same property. Thus $\mathcal{F}$ is a σ-algebra and since every open or closed subset of X is Polish, the preceeding lemma implies that $\mathcal{F}$ contains all open sets. Thus $\mathcal{F}$ contains all Borel set, as desired.

□

Corollary 4.1.20. *If A is an analytic set in a Polish space X, and $B \subseteq A$ is Borel (regarding A as a topological space), then B is analytic.*

Proof. Let Y be a Polish space and $f\colon Y \to A$ a continuous surjective map. Then $f^{-1}(B)$ is a Borel subset of Y, hence by the above theorem we can find a Polish space P and a bijective continous map $\phi\colon P \to f^{-1}(B)$. Then $f \circ \phi$ is a continuous map from P onto B. □

Lemma 4.1.21. *Let X and Y be seperable metric spaces, and let $f\colon X \to Y$ be a Borel map. Then the graph of f is a Borel subset of $X \times Y$.*

Proof. Let U_n be a countable basis for the topology of Y. Then for $y \in Y, x \in X$ we have that $y \neq f(x)$ if and only if there is some n such that $y \in U_n$ and $f(x) \notin U_n$. Thus

$$\{(x, y) : y \neq f(x)\} = \bigcap_{n=1}^{\infty} X \setminus f^{-1}(U_n) \times U_n$$

and since f is a Borel this is a Borel set in $X \times Y$. The above set is the complement of the graph of f, so we are done. □

Corollary 4.1.22. *Let A be an analytic set and let Y be a Polish space. If $f\colon X \to Y$ is Borel, then $f(X)$ is an analytic set in Y. If, in addition, f is injective, then $f(X)$ is Boerl and f is a Borel isomorphism from X to $f(X)$.*

Proof. By the above lemma, the graph of f is a Borel subset of $X \times Y$, and since $X \times Y$ is analytic, it follows that the graph Γ of f is analytic by Corollary 4.1.20. Let $\pi_2\colon X \times Y \to Y$ be the projection onto the second coordinate. Then $\pi_2(\Gamma) = f(X)$, since π_2 is continuous it follows that $f(X)$ is analytic. Suppose now that f is injective. If $B \subseteq Y$ is Borel, then the above implies that $f(B), f(B^c)$ are analytic and since f is injective,

$$f(B^c) = f(X) \setminus f(B).$$

Thus $f(B)$ is a Borel subset of $f(X)$ (regarding $f(X)$ as a topological space) by Corollary **??**. This implies that f is a Borel isomorphism from X onto $f(X)$. □

Corollary 4.1.23. *Let X be a Polish space and let B be a Borel subset of X. Give B the σ-algebra of Borel subsets of B (using the topology of B). Then B is a standard Borel space.*

Proof. By Theorem 4.1.19 there is a Polish space P and a continuous bijective map $\phi\colon P \to B$, the above corollary shows that ϕ is a Borel isomorphism, and the proof is complete. □

Lemma 4.1.24. *Let X be a Polish space with a countable basis of open and closed sets. Then there is collection $\mathcal{C}$ of finite sequence of positive integers, such that $\sigma\big|_k \in \mathcal{C}$, for all $\sigma \in \mathcal{C}$ and $k < |\sigma|$ and a continuous bijection from $T = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma\big|_k \in \mathcal{C}$ for all $k\}$ onto X.*
[11]

Proof. Let d be a compatible complete metric on X. We construct a collection $\mathcal{C}$ of finite integer sequences, such that for each $\sigma \in \mathcal{C}$ there is $N_\sigma \in \mathbb{N} \cup \{\infty\}$ and U_σ a non-empty closed and open set in X such that:

(i) $\varnothing \in \mathcal{C}, U_\varnothing = X$

(ii) If $\sigma \in \mathcal{C}$, then $\sigma 1, \ldots, \sigma N_\sigma \in \mathcal{C}$ and $U_\sigma = \bigcup_{j=1}^{N_\sigma} U_{\sigma j}$

[11]The attentive reader will notice that $\mathcal{C}$ has the structure of an infinite tree, and we are showing that there is a continuous bijection from the set of infinite branches of this tree and X.

(iii) $\operatorname{diam}(U_\sigma) \leq \frac{1}{|\sigma|}$.
(iv) If $j \neq k$ and $1 \leq j, k \leq N_\sigma$ then $U_{\sigma j} \cap U_{\sigma k} = \varnothing$
(v) If $\sigma \in \mathcal{C}$, then $\sigma\big|_k \in \mathcal{C}$ for all $k < |\sigma|$.

We construct $\mathcal{C}$ inductively, by first construct $\mathcal{C}_k$ all of its elements of length at most k. It is clear how to construct $\mathcal{C}_0$. Suppose that $\mathcal{C}_k$ has been constructed so that $(i), (iii), (v)$ are satisfied for all $\sigma \in \mathcal{C}_k$ and $(ii), (iv)$ are satisfied for all $\sigma \in \mathcal{C}_k$ with $|\sigma| < k$. For each $\sigma \in \mathcal{C}_k$ of length k, because X has a countable basis of closed and open sets, we can find a countable cover $V_{\sigma n}$ of closed and open sets contained in U_σ such that $\operatorname{diam}(V_{\sigma n}) \leq \frac{1}{k+1}$. For all n. Let

$$W_{\sigma 1} = V_{\sigma 1}, W_{\sigma j} = V_{\sigma j} \setminus \left(\bigcup_{i=1}^{j-1} V_{\sigma i} \right)$$

then $W_{\sigma 1}$ are now disjoint open and closed subsets of U_σ, which cover U_σ. Set N_σ to be the number of j such that $W_{\sigma j}$ is not empty. Let $U_{\sigma 1}, U_{\sigma 2}, \ldots, U_{\sigma N_j}$ be an ordering of the $W_{\sigma j}$ when $W_{\sigma j}$ is not empty. Set

$$\mathcal{C}_{k+1} = \mathcal{C}_k \cup \bigcup_{\sigma \in \mathcal{C}_k, |\sigma| = k, 1 \leq j \leq N_\sigma} \{U_{\sigma j}\}.$$

It is easy to see that $\mathcal{C}_{k+1}$ satisfies $(i) - (v)$. Having define $\mathcal{C}_k$ for all k, set $\mathcal{C} = \bigcup_{k=1}^\infty \mathcal{C}_k$. Now let $F = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma\big|_n \in \mathcal{C} \text{ for all } n\}$, it is clear that F is closed. By completeness of X, we have a well-defined map $\phi \colon F \to X$ by saying that

$$\phi(\sigma) \in \bigcap_{n=1}^\infty U_{\sigma\big|_n}.$$

As in Theorem 4.1.12 we have that ϕ is continuous and surjective, and (iv) guarantees that ϕ is injective.

□

Theorem 4.1.25. *Let X, Y be Polish spaces and $B \subseteq X$ a Borel set. Let $f \colon X \to Y$ be an injective Borel map. Then $f(B)$ is Borel.*

Proof. Replacing X by $X \times Y$, B by the graph Γ of f, and f by $\pi_2\big|_\Gamma$[12], we may assume that f is continuous. By Lemma 4.1.24 and Theorem 4.1.19 we may assume that $X = \mathbb{N}^{\mathbb{N}}$ and that B is a closed set. For each finite sequence of positive integers σ we will construct a Borel subset B_σ of Y such that

(i) $f\left(\{\sigma' \in B : \sigma'\big|_{|\sigma|} = \sigma\} \cap B\right) \subseteq B_\sigma \subseteq \overline{\{\sigma' \in B : \sigma'\big|_{|\sigma|} = \sigma\} \cap B}$.
(ii) $B_{\sigma|k} \subseteq B_\sigma$ if $k < |\sigma|$.
(iii) If $|\sigma| = |\sigma'|$, and $\sigma \neq \sigma'$ then $B_\sigma \cap B_{\sigma'} = \varnothing$.

Suppose we have constructed such a collection of Borel sets. Set

$$C = \bigcap_{n=1}^\infty \bigcup_{|\sigma| = n} B_\sigma.$$

Then C is Borel, and we claim that $f(C) = B$. Let $\sigma \in B$, then by (i) we have that $f(\sigma) \in B_{\sigma\big|_k}$, and so $f(A) \subseteq C$. Conversely, suppose $y \in C$, because of (ii) and (iii) we can find $\sigma \in \mathbb{N}^{\mathbb{N}}$ such that $y \in B_{\sigma\big|_n}$ for every n. Because of (i), we can find for

[12]Here π_2 is projection onto the second coordinate

each n, a $\sigma_n \in \{\sigma' \in B : \sigma'\big|_n = \sigma\} \cap B$ such that $d(\sigma_n, \sigma) < 2^{-n}$. Since $\sigma_n \to \sigma$ and B is closed, we have that $\sigma \in B$. Thus $y \in f(B)$.

To show that such a collection B_σ exists, let $B_\varnothing = \overline{f(B)}$. Suppose B_σ has been defined for all $|\sigma| \leq k$. Let σ be a sequence of positive integers of length k, set

$$A_n = \{\tau \in B : \tau\big|_k = \sigma, \tau_{k+1} = n\}.$$

Since f is injetive and continuous, the $f(A_n)$ are pairwise disjoint analytic sets, thus by Theorem 4.1.16 we can find disjiont Borel sets $B'_{\sigma n}$ such that $f(A_n) \subseteq B_{\sigma n'}$ for all n. Setting

$$B_{\sigma n} = B_\sigma \cap \bigcap_{n=1}^{\infty} B'_{\sigma n} \cap \bigcap_{n=1}^{\infty} \overline{f(A_n)}$$

to see that $B_{\sigma n}$ has the desired properties. □

We are closed to the proof of our measurable selection theorem, and the measurablitiy of analytic sets. Next we will show the uniqueness of uncountable standard Borel spaces, this will be crucial in the proof of our measurable selection theorem. But first we need a lemma.

Lemma 4.1.26. *Let X, Y be standard Borel spaces and suppose that $f\colon X \to Y, g\colon Y \to X$ are Borel injections. Then there is a Borel isomorphism $h\colon X \to Y$.*

Proof. (From [2] Theorem 1.2.3.) If the reader has read the proof of the Cantor-Schroder Bernstein theorem, we are just going to copy it down. If not, then read on.

We claim that there is a Borel set $E \subseteq X$ such that

$$g^{-1}(X \setminus E) = Y \setminus f(E).$$

Assuming that E exists, we can define $h\colon X \to Y$ by $h(x) = f(x)$ for $x \in E$ and $h(x) = g^{-1}(x)$ otherwise. Then Corollary 4.1.22 implies that h is a Borel isomorphism from X to Y.

For $A \subseteq X$, define $H(A) = X \setminus g(Y \setminus f(X))$, then $A \subseteq B$ implies $H(A) \subseteq H(B)$. Further we have

$$H\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} H(A_n).$$

Indeed, by what we just claim we have

$$H\left(\bigcup_{n=1}^{\infty} A_n\right) \supseteq \bigcup_{n=1}^{\infty} H(A_n)$$

and on the other hand, if

$$x \in H\left(\bigcup_{n=1}^{\infty} A_n\right)$$

then $x \notin g\left(Y \setminus f\left(\bigcup_{n=1}^{\infty} A_n\right)\right)$, thus for each n we have that $x \notin g(Y \setminus f(A_n))$. Define B_n inductively by $B_0 = \varnothing$, and $B_{n+1} = H(B_n)$. Inducitvely, we have that $B_n \subseteq B_{n+1}$, and each B_n is Borel by the proceeding Theorem. Set

$$E = \bigcup_{n=1}^{\infty} B_n,$$

then E is a Borel set, and

$$H(E) = \bigcup_{n=1}^{\infty} H(B_n) = \bigcup_{n=2}^{\infty} B_n = E$$

since $B_n \subseteq B_{n+1}$. But $H(E) = E$ precisely entails that

$$g^{-1}(X \setminus E) = Y \setminus f(E),$$

and this completes the proof.

□

Lemma 4.1.27. *Let $\mathcal{C}$ be a collection of finite sequences of positive integers, such that if $\sigma \in \mathcal{C}$, then $\sigma\big|_k \in \mathcal{C}$ for all $k < |\sigma|$, and such that $T = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma\big|_k \in \mathcal{C}$ for all $k\}$, then T is uncountable. Then we can find $l \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \mathcal{C}$ both of length l such that $\{\sigma \in T : \sigma\big|_l = \sigma_i\}$ is uncountable for each i.*

Proof. Since

$$T = \bigcup_{n=1}^{\infty} \{\sigma \in T : \sigma_1 = n\},$$

and T is uncountable, we can find n_1 such that $T_1 = \{\sigma \in T : \sigma_1 = n_1\}$ is uncountable. If we can find $n \neq n_1$ such that$\{\sigma \in T : \sigma_1 = n\}$ is uncountable, we are done. Else for all $n \neq n_1$ we have that $\{\sigma \in T : \sigma_1 = n\}$ is countable. Because

$$T_1 = \bigcup_{n=1}^{\infty} \{\sigma \in T_1 : \sigma_2 = n\},$$

we can find n_2 such that $\{\sigma \in T_1 : \sigma_2 = n_2\}$ is uncountable. If there is $n \neq n_2$ such that

$$\{\sigma \in T_1 : \sigma_2 = n\}$$

is uncountable we are done. If not, then for all $n \neq n_1$, we have that $\{\sigma \in T_1 : \sigma_2 = n_2\}$ is countable. If the lemma is false, then we see that we can continue inductively to find a $\tau \in T$, such that for all k,

$$\{\sigma \in T : \sigma\big|_k = \tau\big|_k, \sigma\big|_{k+1} \neq \tau\big|_{k+1}\}$$

is countable. As

$$T = \{\tau\} \cup \bigcup_{k \in \mathbb{N} \cup \{0\}, \sigma' \in \mathcal{C}, \sigma'\big|_k = \tau\big|_k, \sigma' \neq \tau, |\sigma'| = k+1} \{\sigma \in T : \sigma\big|_{k+1} = \sigma'\}$$

and $\mathcal{C}$ is countable, we see that T is countable as well. This is a contradiction.

□

Lemma 4.1.28. *If X is a uncountable standard Borel space, then we can find a Borel injection $\phi \colon \{0,1\}^{\mathbb{N}} \to X$.*

Proof. By Theorem 4.1.19 and Lemma 4.1.24 and Corollary 4.1.22 we may assume that there is a collection $\mathcal{C}$ of finite sequences of positive integers, such that $\sigma\big|_k \in \mathcal{C}$ for all $k < |\sigma|$ and $\sigma \in \mathcal{C}$, such that $X = \{\sigma \in \mathbb{N}^{\mathbb{N}} : \sigma\big|_k \in \mathcal{C}$ for all $k\}$. We construct, for each finite binary sequence σ, a sequence $\tau_\sigma \in \mathcal{C}$, such that

(i)$|\tau_{\sigma 0}| = |\tau_{\sigma 1}|$ and $\tau_{\sigma 0} \neq \tau_{\sigma 1}$.
(ii)$\tau_{\sigma 0}\big|_{|\tau_\sigma|} = \tau_\sigma$
(iii) $\{\tau \in X : \tau\big|_{|\tau_\sigma|} = \tau_\sigma\}$ is uncountable.

Set $\tau_{\varnothing} = \varnothing$. Suppose we have constructed τ_σ for $|\sigma| \leq k$, satisfying (i) and (iii) and satsisfying (ii) for $|\sigma| < k$. Fix a binary sequence σ of length k, and let

$$\mathcal{C}_\sigma = \{\tau \in \mathcal{C} : \tau\big|_k = \tau_\sigma\big|_k, \text{ for all } k < \min(|\tau|, |\tau_\sigma|)\}.$$

Then $\mathcal{C}_\sigma$ also has the property that $\sigma\big|_k \in \mathcal{C}_\sigma$ if $\sigma \in \mathcal{C}_\sigma$ and

$$T_\sigma = \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau\big|_k \in \mathcal{C}_\sigma \text{ for all } k\} = \{\tau \in X : \tau\big|_{|\tau_\sigma|} = \tau_\sigma\},$$

so T_σ is uncountable. The above lemma implies that we can find $l \in \mathbb{N}$ and $\tau_{\sigma 0}, \tau_{\sigma 1} \in \mathcal{C}_\sigma$, both of length l, such that $\tau_{\sigma 0} \neq \tau_{\sigma 1}$ and

$$\{\tau \in X : \tau\big|_l = \tau_{\sigma j}\}$$

is uncountable for each j. Then $\tau_{\sigma j}$ satisfy (i)-(iii) by construction. This completes the inductive step of the construction.

Having constructed τ_σ for each σ, we have a well defined map

$$\phi \colon \{0,1\}^{\mathbb{N}} \to X$$

given by

$$\phi(\sigma)\big|_{|\tau_{\sigma|_k}|} = \tau_{\sigma|_k}.$$

As in Theorem 4.1.12, we see that ϕ is injective and continuous. This completes the proof. □

Corollary 4.1.29. *Any two uncountable standard Borel spaces are Borel isomorphic.*

Proof. By the above lemma, we see that $\{0,1\}^{\mathbb{N}}$ Borel injects into any uncountable standard Borel space. By Lemma 4.1.26, if we can show any standard Borel space has a Borel injection into $\{0,1\}^{\mathbb{N}}$ we will be done. But any standard Borel space injects into $[0,1]^{\mathbb{N}}$ by Theorem 4.1.12. Suppose we can show that $[0,1]$ Borel injects into $\{0,1\}^{\mathbb{N}}$. It will then follow that $[0,1]^{\mathbb{N}}$ Borel injects into $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$, which is homeomorphic to $\{0,1\}^{\mathbb{N}}$. This will show that any standard Borel space injects into $\{0,1\}^{\mathbb{N}}$. Thus, it suffices to show that $[0,1]$ has a Borel injection into $\{0,1\}^{\mathbb{N}}$. We shall in fact show that there are Borel isomorphic.

Let D be the set of all dyadic rationals in $[0,1]$ and let $A \subseteq \{0,1\}^{\mathbb{N}}$ be the set of sequences which are eventually constant. Let $\phi \colon \{0,1\}^{\mathbb{N}} \to [0,1] \setminus D$ be given by

$$\phi(\sigma) = \sum_{n=1}^{\infty} \frac{\sigma_n}{2^n}.$$

As is well known, ϕ is a homeomorphism. As A and D are countable, we have a Borel bijection $\psi \colon A \to D$. Then if we define $f = \phi$ on $[0,1] \setminus A$ and ψ on A, it is easy to see that f is a Borel isomorphism between $\{0,1\}^{\mathbb{N}}$ and $[0,1]$.

□

We can now show that every analytic set in a standard measure space is measurable. We shall in fact show that this is true in a very strong sense.

Definition 4.1.30. Let $(X, \mathcal{M})$ be a standard Borel space. The σ-algebra of *universally measurable sets* is defined to be

$$\bigcap_{\mu \text{ a } \sigma\text{-finite measure on } X} \overline{\mathcal{M}}_\mu$$

where $\overline{\mathcal{M}}_\mu$ is the completion of $\mathcal{M}$ with respect to μ. Elements in this σ-algebra are called *universally measurable*. A function $f\colon X \to Y$, where Y is a measurable space is called *universally measurable*, if it is measurable with respect to the σ-algebra of universally measurable sets.

Theorem 4.1.31. *Let X be a standard Borel space, then every analytic set in X is universally measurable.*

Proof. By the above theorem we may assume that $X = [0,1]$, let A be an analytic set in $[0,1]$ and μ a σ-finite measure on X. Then we can find a measure ν on $[0,1]$ which is finite and has the same measure zero sets. Thus we may assume that μ is finite. Let P be a Polish space and $g\colon P \to A$ a continuous surjective map, by Theorem 4.1.12, we may assume that P is a subset of $[0,1]^{\mathbb{N}}$. Let Γ be the graph of g in $P \times [0,1]$, this is a closed subset of $P \times [0,1]$ and is thus a Polish space. Let $Y = [0,1]^{\mathbb{N}} \times [0,1]$, then Γ is a subset of Y. Let $\pi\colon Y \to [0,1]$ be given by $\pi((t_n)_{n\in\mathbb{N}}, x) = x$, then $\pi(\Gamma) = A$. Since Γ is a Polish subset of Y, it is G_δ by Theorem 4.1.13, so we can find (U_n) decreasing and open in Y such that

$$B = \bigcap_{n=1}^{\infty} U_n.$$

Since each U_n is open in Y, and Y is compact metric, for each n we can find an increasing sequence compact sets $K_{n,m}$ such that

$$U_n = \bigcup_{m=1}^{\infty} K_{n,m},$$

and set $K_{00} = X$. Let μ^* be the outer measure on $[0,1]$ associated to the finite measure μ, i.e.

$$\mu^*(E) = \inf\{\mu(U) : U \supseteq E, U\text{open}\}.$$

We shall show that

$$\mu^*(A) = \sup\{\mu(K) : K \subseteq A \text{ compact }\}. \tag{1}$$

Suppose that (1) is shown. Then we can find $K_n \subseteq A$ compact such that $\mu(K_n) \to \mu^*(A)$. Set $F = \bigcup_{n=1}^{\infty} K_n$, by standard measure theory there is a $G_\delta \supseteq A$ such that $\mu(G) = \mu^*(A)$. Thus $F \subseteq A \subseteq G$ and $\mu^*(G \setminus F) = 0$, so A is μ-measurable. So it suffices to show that 1 holds.

Fix $\alpha < \mu^*(A)$. We inductively define integers $j_0 j_1, \dots, j_n, \dots$ such that with

$$C_n = \Gamma \cap \bigcap_{i=1}^{n} K_{i,j_i},$$

we have $\mu^*(f(C_n) > \alpha$. Set $j_0 = 0$, since $K_{00} = X$ this clearly satisfies the construction.Suppose we have constructed $j_0, \dots, j_n$ with this property. Then, since $C_n \subseteq \Gamma \subseteq U_{n+1}$, we have

$$C_n = \bigcup_{j=1}^{\infty} C_n \cap K_{n+1,j},$$

because $K_{n,j}\subseteq K_{n,j+1}$. Note that if A_j are increasing sets in $[0,1]$, then $\mu^*\left(\bigcup_j A_j\right)=\lim_j \mu^*(A_j)$, indeed we can find B_j G_δ sets such that $B_j\supseteq A_j$ and $\mu(B_j)=\mu^*(A_j)$. Replacing B_j with $\bigcap_{k=j}^\infty B_k$ we may assume that B_k is increasing, finally if we show B a G_δ set such that $B\supseteq\bigcup_j A_j$ and $\mu(B)=\mu^*\left(\bigcup_j A_j\right)$ and replacing B_j with $B\cap B_j$ we may assume that $\mu\left(\bigcup_j B_j\right)=\mu^*\left(\bigcup_j A_j\right)$. Then

$$\mu^*\left(\bigcup_j A_j\right)=\lim_{j\to\infty}\mu(B_j)=\lim_{j\to\infty}\mu^*(A_j).$$

Applying to our current situation, we have that

$$\lim_j \mu^*(f(C_n\cap K_{n+1,j})=\mu^*(f(C_n))>\alpha$$

so we can find j_n such that $\mu^*(f(C_n\cap K_{n+1,j})>\alpha$ and this completes the inductive step.

Now define $C=\bigcap_{n=1}^\infty C_n$. Since

$$\bigcap_{n=1}^\infty K_{n,j_n}\subseteq\bigcap_{n=1}^\infty U_n=\Gamma$$

we have

$$C=\bigcap_{n=1}^\infty K_{n,j_n}\cap\Gamma=\bigcap_{n=1}^\infty K_{n,j_n}$$

and C is compact. We have that

$$\mu(f(C))=\lim_n\mu\left(f\left(\bigcap_{k=1}^n K_{k,j_k}\right)\right)\geq\lim_n\mu^*(f(C_n))\geq\alpha.$$

As $f(C)$ is compact, and α is arbitrary, this verifies (1) and the proof is complete. □

Corollary 4.1.32. *Let A be an analytic set and μ a σ-fintie measure on A. Then (A,μ) is a standard measure space.*

Proof. By assumption A is an analytic set in a Polish space X. The above Theorem tells us we can find a Borel $B\subseteq A$ such that $\mu(A\setminus B)=0$. Since a Borel subset of a Polish space is a standard Borel space by Corollary 4.1.23, we have that B is a Borel space and we are done. □

Finally we prove our measurable selection theorem.

Theorem 4.1.33. *Let X and Y be analytic sets, and let $f\colon X\to Y$ be a surjective Borel map. Then there is a universally measurable $\phi\colon Y\to X$ (in the sense that $\phi^{-1}(B)$ is universally measurable for all $B\subseteq X$ Borel*[13]*) such that $f\circ\phi=\mathrm{Id}$.*

Proof. Let Γ be the graph of f in $X\times Y$, which is a Borel set in the analytic space $X\times Y$ by Lemma 4.1.21. Thus by Theorem 4.1.12, we can find a surjective $g\colon\mathbb{N}^{\mathbb{N}}\to\Gamma$, let $h=\pi_2\circ g$, where π_2 is projection onto the second coordinate, then h is continuous. Thus for each $y\in Y$ we have that $h^{-1}(\{y\})$ is a closed set. Let $\prec$ denote lexicographic order on $\mathbb{N}^{\mathbb{N}}$. Note that if $F\subseteq\mathbb{N}^{\mathbb{N}}$ is closed, then F has a

[13] Borel for the topology of X.

least element for $\prec$. Indeed, if we define n_1 to be the least first coordinate among elements of F, and having define $n_1, \ldots, n_{k-1}$ we define n_k to be the smallest k^{th} coordinate among $\sigma \in F$ such that $\sigma\big|_{k-1} = (n_1, \ldots, n_{k-1})$, then since F is closed we have that $n = (n_1, n_2, \ldots) \in F$. So define $\psi(y)$ to be the smallest element in $h^{-1}(\{y\})$ for each $y \in Y$, and let $\phi = \pi_1 \circ g \circ \psi$. Then $f \circ \phi - \mathrm{Id}$, and we only have to show that ϕ is universally measurable. Since g and π_1 are continuous, it suffices to show that ψ is universally measurable. A finite sequence σ of positive integers, let $N_\sigma = \{\tau \in \mathbb{N}^{\mathbb{N}} : \tau\big|_k = \sigma\}$. Then if $\sigma = (n_1, \ldots, n_k)$ we have

$$N_\sigma = \{\tau \in \mathbb{N}^{\mathbb{N}} : (n_1, \ldots, n_k, 1, 1, \ldots) \preceq \tau \prec (n_1, \ldots, n_{k-1}, n_k + 1, 1, 1, \ldots)\}.$$

But for any $\sigma' \in \mathbb{N}^{\mathbb{N}}$ we have that

$$\psi^{-1}(\{\tau \in \mathbb{N}^{\mathbb{N}} : \tau \prec \sigma'\}) = h(\{\tau \in \mathbb{N}^{\mathbb{N}} : \tau \prec \sigma'\}),$$

and similarly for $\preceq$. Thus $\psi^{-1}(N_\sigma)$ is an intersection of two analytic sets and is thus universally measurable by Theorem 4.1.31. Since N_σ clearly generates the topology on $\mathbb{N}^{\mathbb{N}}$, it generates the Borel structure of $\mathbb{N}^{\mathbb{N}}$, and thus ψ is universally measurable.

□

References

[1] Gerald Folland, Real Analyis:Modern Techniques and Their Applications, John-Wiley and Sons, 1999.
[2] S.M. Srivastava, A Course On Borel Sets, Springer, 19991.
[3] Masamichi Takesaki, The Theory of Operator Algebras I, Springer-Verlag,2000.
[4] Masamichi Takesaki, The Theory of Operator Algebras II, Springer-Verlag,2000.