Prof. Thistleton

A wide variety of phenomena may be modelled with a generalization of the exponential distribution. For example, suppose you were interested in the yearly incomes of families who reside in your county. The continuous distributions we have explored so far- the uniform and exponential, are clearly poor choices. We can generalize the exponential distribution, however, to model our data.

Start with an exponential distribution with $\theta = 1$, i.e.

$$g(x) = \frac{1}{\theta} e^{-x/\theta} = e^{-x}$$

"Tie down" the left side by multiplying $g(x)$ by x.

$$f(x) = xe^{-x}$$

Sketch this function and show that this is a valid probability distribution.

Now multiply $g(x)$ by x^2 instead of x. Is this a valid probability distribution? Can we fix this?

Now develop a formula to generate probability distributions of the form

$$f(x) = Ax^n e^{-x}$$

What we have done is to construct a type of distribution called a gamma distribution. In its most general form this distribution is characterized by parameters α and β. To see why these are called gamma distributions consider the following function, called a gamma function

$$\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1} e^{-x} dx$$

This function has several interesting properties. First, evaluate $\Gamma(1)$, $\Gamma(2)$, $\ldots \Gamma(n)$. Also, relate the integral defining $\Gamma(1/2)$ to the integral $\int_0^\infty e^{-x^2} dx$.

We can now build a proper probability density function using these ideas. Note that

$$\int_0^\infty x^{\alpha-1}e^{-x}dx \equiv \Gamma(\alpha)$$

and so we can get a pdf by defining $f(x)$ as

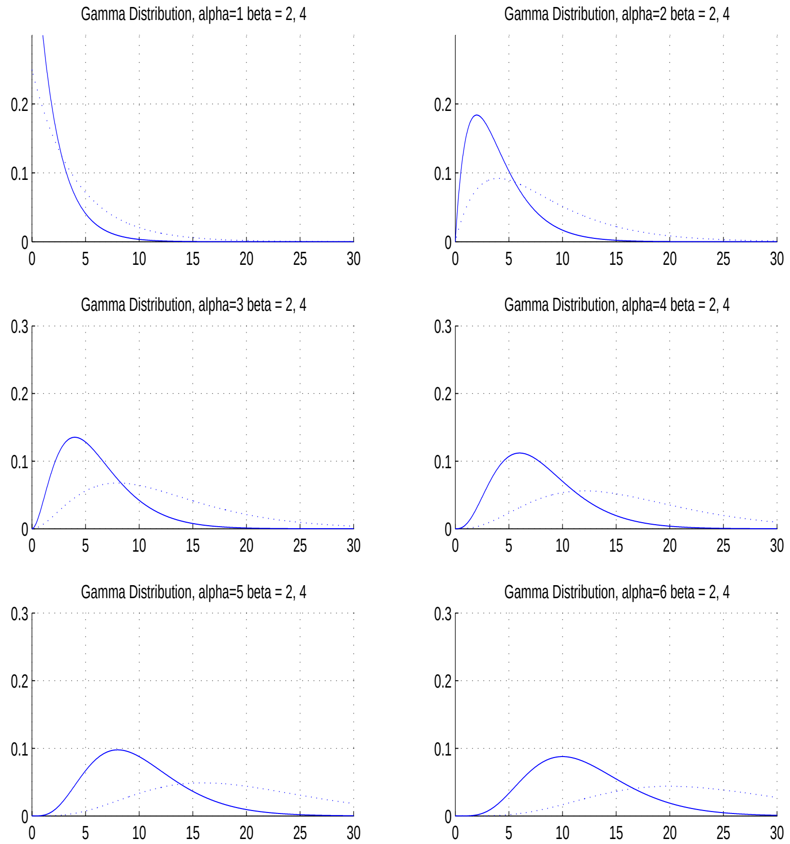
$$f(x) = \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}$$

This is the gamma distribution with parameter α. In order to introduce the other parameter, β, first evaluate the integral

$$\int_0^\infty x^{\alpha-1}e^{-x/\beta}dx$$

Using the answer from above define the gamma distribution with parameters α and β as

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$$



Mean and Variance of a Gamma Distribution Calculate the mean of a gamma distribution with parameters α and β.

Calculate the variance of a gamma distribution with parameters α and β.

Examples One of the reasons why the gamma distribution is so important has to do with the following property: Suppose that X_1 and X_2 are independent, identically distributed (iid) random variables which have gamma distributions with parameters α and β. (We will be more formal about this later). Then $X_1 + X_2$ s also a gamma random variable with parameters 2α and β. We can generalize to the case where X_i $i = 1, 2, \dots, n$ are all iid gamma with parameters α and β. Then

$$X = \sum_{i=1}^{n} X_i$$

has a gamma distribution with parameters $n\alpha$ and β.

1. Suppose that the lifetimes of two components are exponentially distributed with $\theta = 10$ days. The first is put into operation. As soon as the first fails, the second will be put into operation. What is the probability distribution for the total lifetime of the first component and its backup? If you have many such systems, how long would you expect a given system to function on average?

2. You are standing on line at the bank. You have noticed over time that the amount of time a given customer will take has an exponential distribution with $\theta = 1$ minute. If there are 3 people ahead of you in line, calculate your expected wait. Also calculate the probability that your wait will exceed 7 minutes.