Recall:

1. The number, or probability, associated with an event must be between zero and one. That is $0 \leq P(E) \leq 1$.
2. The number, or probability, associated with the sample space must be one. That is $P(S) = 1$.
3. If events A and B do not overlap, that is if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

and

- $A \subset B \Rightarrow P(A) \leq P(B)$.
- $P(A) \leq 1$.
- $P(\emptyset) = 0$.

Show that, in general, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card, and let B be the analogous event for a MasterCard. Suppose that

- $\text{prob}(A) = 0.5$
- $\text{prob}(B) = 0.4$
- $\text{prob}\ (A \cap B) = 0.25$

Calculate the probability that

1. the selected individual has at least one of the two types of cards.
2. the selected individual has neither type of card.
3. the selected individual has a Visa but not a MasterCard.
4. the selected individual has exactly one card.

Conditional Probability:

A natural question to ask is whether knowing an event has occurred gives you any information about whether another event might occur. For example, if you have a set of unmarked keys and wish to open a door, you will (hopefully) note which keys don't work as you continue to try to get through (sample without replacement).

Definition: Let A and B be two events associated with a sample space S. Define the **conditional probability** of B given A as

$$P(B|A) \equiv \frac{P(A \cap B)}{P(A)}$$

as long as $P(A) \neq 0$.

The idea to keep in mind here is that we work with a reduced sample space when considering conditional probabilities.

Again working with the Visa and MasterCard example, calculate the probability that
Calculate the probability that:

1. the selected individual has a Visa card given that she has a MasterCard.
2. the selected individual has a MasterCard given that she has a Visa card.
3. the selected individual does not have a MasterCard given that she has a Visa card.
4. the selected individual does not have a MasterCard given that she does not have a Visa card.

According to the American Lung Association (http://www.lunusa.org) 27.6% of American adult men are smokers, as are 22.1% of American adult women. If we assume that 52% of American adults are women, what is the probability that a randomly chosen American adult will be a smoker? Suppose all you know about an adult is that they smoke. What is the probability that they are male? female?

Suppose you wish to test a population for a certain characteristic. For instance, you may wish to test high school athletes for the use of steroids. Unfortunately, tests may be imperfect. They may produce false positives or false negatives. In our case a false positive corresponds to a “clean” athlete who tests positive for steroid use. (What would a false negative be?)

We’ll need two definitions: the true negative rate of a test is called its specificity. The true positive rate is called its sensitivity. Suppose you are administering a test with specificity 98% and sensitivity 90%. You already know from other sources that the prevalence of steroid use among student athletes is 1%.

You test an athlete and find that the test indicates steroid use. What is the probability that the athlete has, in fact, used steroids?

Independence: When conditioning provides no information on probabilities two events are said to be *independent*. That is, A is said to be independent of B if

$$P(A|B) = P(A)$$

Show that if A is independent of B then B is independent of A and we may just speak of the independence of two events.

Show that if A and B are independent then $P(A \cap B) = P(A)P(B)$.

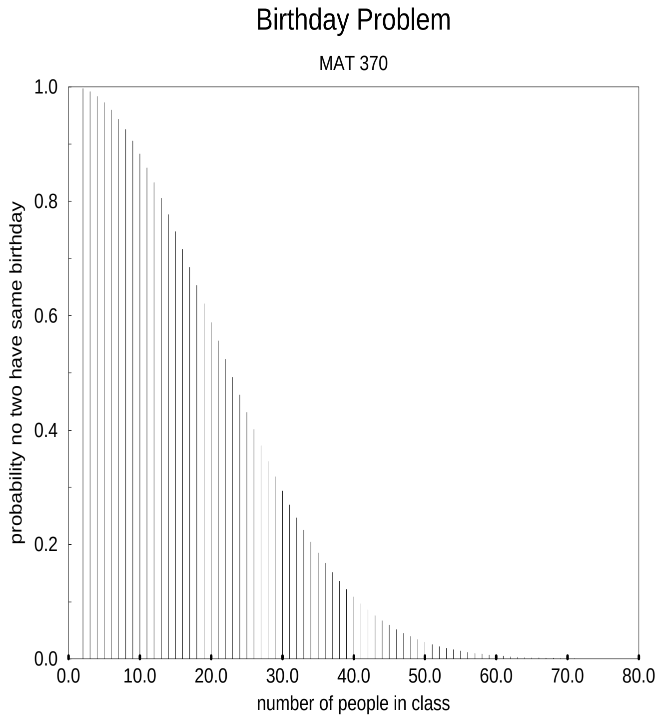
When there are more than two events under consideration we need to make a distinction between pairwise independence (weaker) and mutual independence (stronger). Define a collection of sets $A_1, \ldots, A_n$ to be mutually independent if, for any collection of these events $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ we have

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdot \ldots \cdot P(A_{i_k})$$

Note that pairwise independence does not guarantee mutual independence.

Example: Toss a fair coin 6 times. What is the probability of obtaining 3 heads?

Example: There are 24 people in a room. What is the probability that at least two of them have the same birthday?



Bayes' Theorem: We often search for the *cause* of an event. In the case where the sample space is partitioned into events B_1, B_2, $\ldots$, B_n we may calculate the probability that event B_j is the cause of event A. That is, calculate $P(B_j|A)$.

The Theorem:

$$P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}$$

proof

Example: A bolt factory produces bolts on three production lines: A, B, and C. Line A accounts for 25% of the bolts produced, line B for 35%, and line C for 40%. On line A 5% of the bolts produced are defective, on line B 4% are defective, and on line C 2% are defective. You have a defective bolt. What is the probability that the bolt came from line A? From line B? From line C?