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Consider random variables defined with non rectangular support. Define a joint density as follows:

$$f(x,y) = \begin{cases} ky(1-x-y) & 0 < x, 0 < y, x+y < 1 \\ 0 & else \end{cases}$$

Calculate the marginal distributions $f_X(x)$ and $f_Y(y)$ and calculate the covariance of X and Y. Also, define $Z = X + Y$ and calculate the PDF of the univariate random variable Z.

As an interesting example, define $Y = F(X)$ for a random variable X where $F(x)$ is the cumulative distribution function of X. What is the PDF of Y?

Recall: *Conditional Distributions*

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \quad f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

Consider the following example taken from our text: Suppose that two components of a minicomputer have the following joint PDF for their useful lifetimes X and Y.

$$f(x,y) = \begin{cases} xe^{-x(1+y)} & 0 < x, 0 < y \\ 0 & else \end{cases}$$

Calculate the probability that the lifetime of the first component, X, exceeds 5 years given that the second component lasted exactly 3 years, i.e. $Y = 3$.

Transformations of Random Vectors Recall that the PDF of $Y = u(X)$ may be found, under the relatively common assumptions that u is strictly monotone and also differentiable, as

$$f_Y(y) = f_X(u^{-1}(y))|\frac{du^{-1}(y)}{dx}|$$

This approach may be extended to random vectors as well. As a motivating example, suppose that X_1 and X_2 are independent, identically distributed exponential random variables with expected value equal to one, i.e. $X_1, X_2 \overset{iid}{\sim} exp(1)$. Define $Y_1 \equiv X_1 + X_2$ and $Y_2 \equiv X_1/(X_1 + X_2)$.

Our goal: to find the joint pdf of (Y_1, Y_2) and the marginal distributions of Y_1 and Y_2. First we'll need to establish some notation.

$$\begin{aligned} Y_1 &= u_1(X_1, X_2), \quad Y_2 = u_2(X_1, X_2) \\ X_1 &= w_1(Y_1, Y_2), \quad X_2 = w_2(Y_1, Y_2) \end{aligned}$$

Then we'll need the following definition. Define the *Jacobian* of our transformation as the determinant of the 2×2 matrix of partial derivatives

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$

The key result, which may be extended to higher dimensions in a direct manner, is as follows. Note the similar structure of the formula to those obtained above in the single variable case.

$$f_{\underline{Y}}(y_1, y_2) = f_{\underline{X}}(w_1(y_1, y_2), w_2(y_1, y_2))|J|$$

where $|J|$ denotes the absolute value of the Jacobian. Apply this result to our problem.

***Moment Generating Functions**:* These functions will be very convenient tools and computational aides as we work with both continuous and discrete random variables. In particular, once we have computed the moment generating function for a random variable, the calculations of that random variable's mean and variance are greatly simplified. We will also make use of these functions to calculate the probability distributions of functions, especially sums, of random variables.

Denote by $M_X(t)$ the moment generating function of a random variable X and define this function as

$$M_X(t) \equiv E(e^{tX})$$

Recall that for discrete distributions with probability mass function $f(x_i)$

$$E(g(X)) = \sum_{x_i} g(x_i) f(x_i)$$

Therefore

$$E(e^{tX}) = \sum_{x_i} e^{tx_i} f(x_i)$$

Similarly, for continuous RV's

$$E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Calculating Means: We can use the previously calculated moment generating functions to calculate the mean and variance (in fact, all the moments) of the corresponding random variables as follows. Take a first derivative of $M_X(t)$ with respect to t (assuming that we can bring the derivative into the summation)

$$\begin{aligned}
\frac{d}{dt}M_X(t) &= \frac{d}{dt}\sum_{x_i} e^{tx_i}p(x_i) \\
&= \sum_{x_i} \frac{d}{dt}e^{tx_i}p(x_i) \\
&= \sum_{x_i} x_i e^{tx_i}p(x_i)
\end{aligned}$$

Now evaluate the derivative at $t = 0$.

$$M_X'(0) = \sum_{x_i} x_i e^{0x_i}p(x_i) = \sum_{x_i} x_i p(x_i)$$

We see that the first derivative of the moment generating function evaluated at $t = 0$ is the expectation of the random variable! This result holds in the continuous case as well.

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Calculating Variances: Recall that we may compute the variance of a random variable as

$$V(X) = E(X^2) - E(X)^2$$

Take a second derivative of the moment generating function

$$\begin{aligned}\frac{d^2}{dt^2}M_X(t) &= \frac{d}{dt}\sum_{x_i} x_i e^{tx_i} p(x_i) \\ &= \sum_{x_i} x_i^2 e^{tx_i} p(x_i)\end{aligned}$$

and so

$$M_X''(0) = \sum_{x_i} x_i^2 e^{0x_i} p(x_i) = E(X^2)$$

Examples:

1. *Gamma Random Variables*

2. *Normal Random Variables*

Reproductive Property of Gamma Random Variables
Using moment generating functions, show (for what conditions?) that the sum of two independent Gamma random variables is also a Gamma Random Variable.

(Reproductive Property of Normal Random Variables)
Using moment generating functions, show that the sum of two independent normal random variables is also a normal random variable.