Inferentialism, Logicism, Harmony, and a Counterpoint

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Abstract

Inferentialism is explained as an attempt to provide an account of meaning that is more sensitive (than the tradition of truth-conditional theorizing deriving from Tarski and Davidson) to what is learned when one masters meanings.

The logically reformist inferentialism of Dummett and Prawitz is contrasted with the more recent quietist inferentialism of Brandom. Various other issues are highlighted for inferentialism in general, by reference to which different kinds of inferentialism can be characterized.

Inferentialism for the logical operators is explained, with special reference to the Principle of Harmony. The statement of that principle in the author's book *Natural Logic* is fine-tuned here in the way obviously required in order to bar an interesting would-be counterexample furnished by Crispin Wright, and to stave off any more of the same.

1 Introduction

In 'Five Milestones of Empiricism', Quine identifies three levels of theoretical focus in semantics: terms, sentences, theories. These are levels of *syntactic and/or set-theoretic containment*. Terms are constituents of sentences, and sentences make up theories. Another sequence of levels, however, can be determined from the increasing *complexity of logico-linguistic functioning* involved: terms, sentences, and *inferential transitions*.[1] For the inferential semanticist, it is the levelling by complexity of logico-linguistic functioning that is theoretically most useful.

Post-Quinean semantics is of two broad kinds:

1. sentence-focused, truth-conditional semantics; and

2. sequent-focused, inferential semantics.

1.1 Sentence-focused, truth-conditional semantics

The truth-conditional theorizing of Tarski and Davidson is well known, and easily the dominant paradigm among contemporary Anglo-American analytical philosophers. Their theorizing is based on the central conceptual link between truth and meaning that is provided by the celebrated *adequacy condition* on one's overall theory of truth and of meaning.[2] The adequacy condition is that the theory should yield every instance of Tarski's famous

> Schema T: s is T if and only if p.

Each instance of this schema is obtained by replacing 's' with a metalinguistic term[3] denoting a sentence of the object-language, and replacing p by a translation of that sentence into the metalanguage (here, taken to be English). Note the focus on *sentences*, as the minimal unit of linguistic communication.

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[1] Inferential transitions are here thought of as the basic steps involved in arguments, or justifications that are based on premises. The logician sometimes formalizes inferential transitions as rules of natural deduction or as *sequent* rules. Of course, it is not being suggested that ordinary speakers would need to know anything about rules of inference in natural deduction—or about sequents and sequent calculi—in order for an inferential semantics for their language to be correct and fruitful.

[2] Note the addition, here, of the words 'and of meaning'. This is in order not to prejudge the issue, about to be explained, of the direction-of-dependence of the notion of truth on that of meaning, or *vice versa*.

[3] Tarski spoke of *structural-descriptive* terms.

It is commonly maintained that Tarski took the notion of translation (i.e., meaning-preserving mapping) as given, and used that notion to ensure that fulfillment of the adequacy condition entailed that the defined predicate T was indeed a truth-predicate. For Tarski, therefore, the conceptual route would be from translation (meaning) to truth, courtesy of the T-schema.

It is also commonly maintained that Davidson [9], [10] reversed the direction of conceptual dependency. Davidson sought observational constraints on the postulation of (conjectural) biconditionals of the Tarskian form. In such postulated biconditionals, the predicate on the left *would already be interpreted as the truth-predicate*. Consequently, the empirically (albeit holistically) confirmed fulfillment of the adequacy condition would entail that the right-hand sides of the Tarskian biconditionals could be taken as (giving) the meanings of the object-language sentences referred to on the left-hand sides. For Davidson, the conceptual route would be from truth to meaning, courtesy of the T-schema. The standard wisdom, therefore, appears to be that if one takes either one of the notions of truth or translation (meaning-specification) as given, then Tarski's adequacy condition delivers the other notion.[4]

Interpretative truth-conditional theorizing in the manner of Davidson might provide a picture of finished, fully acquired competence with a language. It is not, however, a promising account of how the learner of a first language can *acquire* such competence. The theoretical apparatus presupposes finished competence on the part of the theorizer. Moreover, as the project of 'radical interpretation' proceeds, a high degree of logical sophistication is demanded of the theorizer. For the theorizer has to undertake theory-revision, in order to accommodate his theoretical conjectures to bits of now-confirming, now-disconfirming, incoming data. Learners of a first language cannot learn that way; for, in order to do so, they would have had to acquire a language in the first place.

1.2   Sequent-focused, inferential semantics

For the epistemologist of linguistic understanding, it is tempting to look elsewhere for a characterization of what it is that one grasps when one understands logically complex sentences of one's language. The temptation is to try to characterize sentence meanings *via* an account of *how one comes to grasp them*. The idea is to attend closely to the to-and-fro in our

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[4]The present author challenges both of these 'priority-granting' accounts, and proposes instead an account of truth and meaning as *coeval*, even when conformity with the (neutrally stated) adequacy condition is the central goal. See [36].

use of words: not only the to-and-fro of conversation, but also the to-and-fro of *inference*. One infers *to* conclusions, and one infers *from* premises. The meaning of a sentence can be reconceived as based on what it takes to get *to* it; and on what one can take *from* it in getting elsewhere. Semantics in the manner of Tarski and Davidson concerned itself solely with the sentence-world relationship. But perhaps even that relationship can be re-conceived?—as involving, say, *language-entry rules* (for moving from perception to observational reports) and *language-exit* rules (for moving from sentences heard or inferred, to actions).[5] Moreover, once *within* language (whether a publicly spoken one, or a language of thought) there are the moves *among* sentences, beginning at the points of entry and ending at the points of exit. Are not those moves governed by rules sufficiently exigent for us to be able to appraise the moves as right or wrong, as well- or ill-advised?

This is the challenge to which inferentialism, broadly construed, responds. In doing so, one is really returning to source. For it was Frege himself who, in §3 of the *Begriffsschrift*, gave a contextual definition of the *propositional content* of a sentence P as what P has in common with any sentence that features the same way as P does as a premise in valid arguments. Thus, P and Q have the same propositional content just in case:

> for every set Δ of sentences, for every sentence R, the argument $\Delta, P : R$ is valid if and only if the argument $\Delta, Q : R$ is valid.

2 Inferentialism

Inferentialist accounts of meaning vary considerably. One can identify at least five dimensions of variation.

1. Inferentialist accounts may differ on the question of how ambitious or extensive a grounding of normativity and content is claimed to be rendered by the inferential basis proposed. *Strong* inferentialism says 'inferences are the be-all and end-all of semantics', and seeks to explain even the notions of singular reference and (relational) predication in terms of inference, taking the notion of inference to be more fundamental—both for constitution of meanings and for acquisition of grasp of meanings. By contrast, *moderate* inferentialism ventures only so far as to claim that inferences (or *patterns* of inference) deter-

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[5]The original source for the idea of language-entry and language-exit rules is Sellars [26].

mine the meanings of all the logical operators,[6] and perhaps also the meanings of some important mathematical ones. But the moderate inferentialist does not seek to reduce every significant meaning-affecting syntactic distinction to patterns of inference.

2. A second dimension of variation is in response to the question 'How complicated a system of inferences is needed in order to transform mere signalling into fully fledged linguistic practice?' *Holistic* inferentialism will insist on the presence of a very wide set of inferences, or inference-patterns, including ones involving complex premises and conclusions, before being willing to grant linguistic status to the system of signalling, or information-transfer, concerned. By contrast, *molecularist* inferentialism maintains that the meaning-determining patterns of inference (where they apply) are reasonably operator-specific. Operators can be grasped individually, and not necessarily only as part of one big package. So one's theoretical account of meaning-determination should seek to isolate and characterize, for each operator, its central or canonical meaning-determining patterns of inference.

3. A third way in which inferentialist accounts can differ is by taking different stands on the question whether human beings and their linguistic practices form part of a reasonably continuous natural order, or whether there is some 'conceptual Rubicon' that separates us from other animals, making our languages essentially different from other animals' signalling systems. *Naturalist* inferentialism will countenance 'phase transitions' in complexity of psychology and of behavior. But a naturalist account will insist that human linguistic and conceptual abilities must have arisen by natural selection working on various more primitive capacities of our non-linguistic primate ancestors. Our abilities and faculties, remarkable though some of us may take them to be, are still the products of natural, evolutionary processes. By contrast, an anti-naturalist, or *hyper-rationalist* inferentialism insists that human thought and language is unprecedented, unique, and of a totally different order than what obtains in the rest of the animal kingdom; it is an all-or-nothing collection of faculties that cannot be possessed piecemeal, and the likes of which have no plausible evolutionary precursors in lowlier animals.

4. A fourth dimension of variation is whether the inferences to which the

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[6]For good measure we include the identity predicate among the 'logical' notions, so perhaps 'logical operators' should read 'logical operators or predicates'.

inferentialist account in question appeals are (i) those that enable us to evaluate logically complex statements for truth and falsity against a background of atomic facts, or (ii) those that enable us to make deductive transitions from logically complex premises to logically complex conclusions. One might call *evaluative* inferentialism the view that inferences of type (i) suffice for the explanation of meaning and content; while the *global* inferentialist insists on the need to consider inferences of type (ii) as well.

5. A fourth *differentia* of inferentialist accounts is whether classical logic emerges unscathed as the right logic. On *logically quietist* accounts, it does; on *logically reformist* accounts, some *proper subsystem* of classical logic is favoured—usually intuitionistic logic.

6. Finally, inferentialist accounts can differ in how they define formally (deductively) correct inference in terms of materially correct inference, *if* they take materially correct inference as an independently available notion.

Interest in the alternative approach of inferential semantics is growing. Prawitz [23], [24] and Dummett [12], [13], [14], provided the initial impetus to approach the theory of meaning from an inferentialist perspective. Their interest in inferentialism was driven by manifestationist concerns in the theory of meaning, in turn leading to an anti-realist outlook on language, mind and world. Given the particular way they pursued their inferentialism, one important outcome was a *reformist* attitude towards deductive logic. In their view, inferentialism enjoined *intuitionistic* logic as the correct logic for our logical words, once their role in inference is properly characterized. Inferentialists of the Dummett-Prawitz school are skeptical about the graspability of the purported 'classical' meanings of logical operators such as negation, implication and existential quantification.

More recently, however, Brandom [4], [5], [6] has developed a 'pragmatist' version of inferentialism that is significantly different in its details from the accounts that Prawitz and Dummett would favor, especially in eschewing the route to anti-realism.[7]

In terms of the contrast developed above, Brandom's inferentialism is (1) strong, (2) holistic, (3) anti-naturalist/hyper-rationalist, (4) global, and

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[7]The inferentialism of [6] is also a departure from Brandom's earlier [3], in which his main concern was with the contrast between truth-conditional and assertibility-conditional theories of sentence meaning. In the later work [6], the notion of inference occupies center stage.

(5) logically quietist. The inferentialism of Dummett and Prawitz is also (4) global, but, by contrast with Brandom's, is (1) moderate, (2) molecularist, and (5) logically reformist. Issue (3) appears, for Dummett and Prawitz, to be moot.

Brandom shares with Prawitz and Dummett their general methodological conviction: that illumination of semantic matters is best sought by studying patterns of inference among sentences, rather than by attempting to characterize those sentences' truth-conditions. Nevertheless it appears to be the case that, for Brandom, *classical* logic survives intact after his inferentialist re-construal of meaning.

2.1 Brandom's inferentialism and the choice of logic

In his early paper [4] setting out the main ideas behind his inferentialism, Brandom developed a notion of logical consequence by appeal to a relation of 'material incompatibility'. He argued (p. 48) that one of the 'representative validations' is Double Negation [Elimination]: $\neg\neg p \to p$.

In both [19] and [20], Mark Lance and Philip Kremer study a proof theory and algebraic semantics for the Brandomian notion 'commitment to A is, in part, commitment to B', but limit themselves to conjunction and the conditional. At the end of [20] (at p. 448) they write that '[t]here are reasons for trepidation' about extending their account to deal with the interactions of the conditional with disjunction and with negation. They also concede that the [intuitionistic] logical truth $(A \to \neg A) \to \neg A$ 'is hard to motivate on our interpretation of "$\to$".'

In [17], Lance takes double negation elimination as an axiom, in a 'set of uncontroversial principles' (p. 116), without questioning whether an inferentialist semantics would validate it. Lance's central notion of *permissive entailment* is also inherently classical:

> . . . if we let **f** stand for "The Bad," say a disjunction of all the claims whch are untenable in a given context, we see that permissive inference is definable in terms of committive: A permissively entails B (hereafter A⊨B) iff A&∼B committively entails **f** (A&∼B⊢**f**).

It is the 'if' part of this biconditional that the intuitionist would not accept, since it is tantamount to classical *reductio*.

In [18], Lance leaves his reader with no clear indication of whether a Brandomian inferential semantics validates classical logic. He ends (p. 456) with the following rather gnomic passage:

> If we were to require ...that agents be committed to all theorems, as opposed to having the inferential content of their commitments articulated by those theorems, relevance would collapse, and the system would be unsuitable to Brandomian purposes. But this is not to say that there isn't some privileged status had by classical tautologies such as $A \vee \neg A$ ....

Those developing the technical implications of Brandom's inferentialism would appear not to have engaged satisfactorily with—let alone settled—the question whether it validates all of classical logic. By contrast, inferentialists in the Dummett–Prawitz tradition identify the introduction and elimination rules as central, emphasize the importance of an accompanying notion of *direct proof*, or *warrant*, and proceed vigorously to a principled preference for intuitionistic logic.

2.2   Other noteworthy points

The present author finds three other points to be noteworthy when comparing Brandom's inferentialism with that of Dummett and Prawitz.

(i) Despite the current climate of materialist metaphysics and naturalized epistemology, none of these named figures has seriously engaged the problem of how *normative* logical relations might be possible even if one were to opt for the naturalism described under (3) above.

(ii) Dummett and Prawitz, unlike Brandom, are inspired mainly by, and focus mainly on, the language of mathematics, with the consequence that their inferentialism might be regarded as not 'strong enough', in the sense of 'strong' that is described under (1) above.

(iii) Despite his focus on inference, Brandom's account does not engage with the methods, techniques and results of modern proof theory, which, for Dummett and Prawitz by contrast, supplies the main materials by means of which they formulate their reformist case.

Ad (ii): Those who, like Dummett and Prawitz, are inspired mainly by, and focus mainly on, the language of mathematics, can be forgiven if they inadvertently give the impression that their inferentialism is not 'strong enough', in the sense of 'strong' that is described under (1) above. There are at least three ways that a more sympathetic reconsideration of their contribution could defend them against this objection.

First, one could hold, on behalf of Dummett and Prawitz, that the impression of 'moderateness' in this regard derives from the relative absence, in the case of mathematical language, of any 'language-entry' and 'language-exit' rules. Thus their approach could be understood as apt for *extension* so as to be able to handle language-fragments for which such rules become genuinely operative.

Secondly, if this first option be regarded as special pleading, one could hold that, to the extent that there *must be* 'language-entry' and 'language-exit' rules for *any* language-fragment, one needs to attend to the role that is played in the Dummett–Prawitz account by the requirement of *recognition that a given construction is a proof*. This is the analogue, in the case of mathematics, of perception of worldly things. That is to say, it is the analogue of that which is governed by language-entry rules. Moreover, the speech act of *assertion backed by proof*—which is the basic move in successful communication in mathematics—is the analogue of action directed at worldly things. That is to say, it is the analogue of that which is governed by language-exit rules.[8]

Thirdly, one could look for the 'language-entry' and 'language-exit' rules in a slightly different location within the overall intellectual landscape. One could hold that the 'language-entry' rules *for the language of mathematics* are those by means of which one expresses scientific hypotheses in the language of mathematics—hypotheses that would otherwise be expressed purely 'synthetically', without using any mathematical vocabulary. And the 'language-exit' rules (again, *for the language of mathematics*) are those by means of which, after taking advantage of the tremendous deductive compression afforded by the use of mathematics, one traverses back from mathematically expressed consequences of one's scientific hypotheses to those purely synthetic, observational predictions by means of which the hypotheses are empirically tested.[9]

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[8]Sellars's language-exit rules were of course more mundane, involving intentional action within the world, directed towards external things. But remember that we are here seeking, on Dummett's and Prawitz's behalves, some *analogue* of this for the language of mathematics; and the analogy may well have to be stretched. *Showing* one's interlocutor a token of a proof brings to an end the drawing of inferences on one's own part. It is then the interlocutor's turn to recognize the proof for what it is.

[9]The reader familiar with Hartry Field's [15] will recognize the setting. It is that within which Field himself was concerned to demonstrate that mathematical theorizing afforded a *conservative extension* of the synthetic scientific theorizing that is expressible without recourse to mathematical vocabulary.

3   The author's preferred version of inferentialism

In [30] and [34], the present author sought to deepen the inferentialism of Dummett and Prawitz, and to argue for *intuitionistic relevant* logic as the right logic. So the inferentialism developed in those works was (1) moderate, (2) molecularist, (4) global, and (5) logically reformist.

But it was also (3) naturalist. It was found that inferentialism provided a particularly congenial setting in which to pursue the systematic (if speculative) *naturalistic* explanation, begun in [29], [31] and [32], and re-visited in [33], as to how a logically structured language could have evolved from a logically unstructured signalling system. Crucially, the harmony (see §7 below) that an inferentialist says must obtain between the introduction and elimination rules for any logical operator becomes a necessary precondition for the possibility of evolutionary emergence, and subsequent stability, of the operator within an evolving, logically structured language.

This naturalist, evolutionary view was extended further—first in [30], and later in [35]. In [30], an account was given of so-called 'transitional atomic logic', whereby the connectives could be understood in terms of inference rules and derivations. in [35], *material rules of atomic inference* (especially those registering metaphysical contrarieties) were used in order to show how negation (and the resulting notion of the *contradictory* of a proposition) could arise from just a prior grasp of contrarieties. It is those contrarieties that precipitate occurrences of the absurdity symbol, by reference to which one can then frame the introduction rule for negation. And that introduction rule fixes the meaning of negation. (The usual elimination rule is uniquely determined as the harmoniouely balancing companion for the introduction rule.) If that systematic account of the origin of negation within an increasingly complex language holds up, then one cannot escape the essentially *intuitionistic* character of negation.

Brandom, too, has appealed to contrarieties to explain the role of negation. He defines the negation of a proposition as its 'minimal incompatible'. In doing so, he gives essentially the same definition as just described. His definition takes the form of a definite description: $\neg p$ is *the* proposition r such that (i) r is incompatible with p, *and* (ii) r is implied by any proposition incompatible with p (see [5] at p. 115). This definition does not, however, secure anything more than the intuitionistic meaning for negation. Yet Brandom appears to assume that negation, thus defined, must be classical. (See [5], at p. 115: 'It has been shown ... how to represent classical logic ... by constraints on incompatibility relations.' See also note 73 on p. 668.)

Both Dummett and Prawitz confined themselves to logical connectives and quantifiers. These are *sentence*-forming operators. The present author has sought to extend the spirit and methods of their approach so as to deal also with *term*-forming operators. It turns out that such operators, too, can be furnished with carefully crafted introduction and elimination rules. And these rules arguably capture the 'constructive content' of the notions involved—such as 'number of' and 'set'. Foremost among these are the definite description operator, the set-abstraction operator, and the number-abstraction operator. Identifying the right introduction and elimination rules for these operators is an interesting challenge, and one that, when met, affords a clear distinction between the *analytic* part of (say) set theory, and its *synthetic* part. See [37] for details.

Those of an analytic or logicist bent might find this general kind of approach attractive, for it furnishes a principled way of distinguishing that part of a theory that can be said to be analytic from the part that should be conceded to be synthetic. (See the discussion in [34], ch. 9 for more details.) Roughly, the analytic part is generated by the introduction and elimination rules. Analytic results involve either no existential commitments at all, or commitments only to necessary existents. Ironically (in this context) the best example of an analytic portion of a theory is what Quine, the great opponent of the analytic/synthetic distinction, himself called 'virtual' set theory. This is the part of set theory that is free of any existential commitments. (It is precisely this part that is captured by the introduction and elimination rules for the set-abstraction operator.) Furthermore, the analytic results can always be obtained constructively, since one has no truck with classical rules when applying only introduction and elimination rules.

4 Constructive logicism and Wright's neo-Fregean logicism

In [30] the present author developed the doctrine of 'constructive logicism', as the anti-realist's inferentialist re-working of Crispin Wright's resuscitation (see [42]) of Frege's treatment of natural numbers as logical objects. Wright had sought to begin his derivations from further downstream within the *Grundgesetze*, by using Hume's Principle:

$$\#xF(x) = \#xG(x) \iff \exists R \ \ F \xleftrightarrow[R]{\text{1-1}} G \ ;$$

as his starting point. But all the while Wright used standard, *unfree*, classical logic, despite the presence, in the language, of number-abstraction terms of the form $\#xF(x)$ ('the number of Fs').[10] By contrast, the present author took the challenge to be that of logicizing arithmetic *within the self-imposed constraints of anti-realist inferentialism*. The aim was to derive the Peano–Dedekind axioms for pure arithmetic using *logically more fundamental* principles than Hume's Principle, and to do so within the system of intuitionistic relevant logic.

Any concern for constructivity was puzzlingly absent from Wright's [42]. Puzzling also was the casual-seeming acceptance of the 'universal number' $\#x(x = x)$ (*aka* 'anti-zero') whose existence was entailed by Wright's adoption of an unfree logic for such abstraction terms. In [30] (at p. 236) the present author raised an objection against Wright's too-ready acceptance of the universal number.[11] That objection was raised later by Boolos [2], apparently independently, and has since come to be known as the 'Bad Company' objection. The subsequent arguments that Wright has advanced for the analyticity of Hume's Principle also cause one to ask whether he might not have been better off identifying logically more fundamental meaning-conferring rules than Hume's Principle, in order to advance his case that much arithmetical knowledge is analytic. For, to reiterate a point made above: analytic results can always be obtained constructively, since one has no truck with classical rules when applying only introduction and elimination rules.[12]

The logically more fundamental principles advocated in [30] were *meaning-constituting rules for the introduction and elimination* of the arithmetical expressions 0, $s(\ )$, $\#x(\ldots x \ldots)$ and $N(\ )$. The rules had to be stated within a free logic, since one needed to be able to detect exactly where one's existential commitments to the numbers crept in.[13] Note, however, that addition and multiplication were outside the scope of the treatment, as indeed they had been for Wright himself (and, even if somewhat surprisingly, for Frege himself, as is argued in detail in [40]).

The basic adequacy constraint that [30] imposed on a logicist theory of

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[10]Wright used 'N' for '#'.

[11]Happily, Wright has since disavowed the universal number, and indeed sought its 'exorcism'—see Essay 13 in [16], at pp. 314–5. Philosophy, one is all too often reminded, is a constant battle against the bewitchment of the intellect by language.

[12]For this reason, the present author has more recently pursued a study of Lewisian mereology, in order to show that Lewis's important conceptual argument for his Second Thesis in [21] can be constructivized, once one has the right system of introduction and elimination rules for the fusion operator in mereology. See [41] for details.

[13]The best statement of the rules of constructive logicism can be found in [37].

arithmetic was the derivability of every instance of

Schema N: $\#xF(x) = \underline{n} \leftrightarrow \exists_n xF(x).$[14]

This is to be read as: 'The number of Fs is identical to $\underline{n}$ if and only if there are exactly n Fs'. For example, for $n = 2$ the relevant instance of Schema N is

$$\#xFx = ss0 \leftrightarrow \exists x \exists y (\neg x = y \wedge Fx \wedge Fy \wedge \forall z (Fz \rightarrow (z = x \vee z = y))).$$

The original version of constructive logicism developed in [30] has since been extended, in [40], to an inferentialist account that deals not only with zero, successor, 'the number of ...' and '... is a natural number', but also with the operations of addition and multiplication. These two operations receive inferentialist treatment by employing the notion of *orderly pairing*. The main virtue of this full-fledged inferentialist treatment of arithmetic is that it meets a stringent Fregean requirement that one explain how numerical terms (including additive and multiplicative ones) find application in counting finite collections. The employment of the notion of orderly pairing, and its correlative notions of 'left projection' and 'right projection', brings to light an interesting feature of these (arguably, logical) notions: all three functions—pairing, projecting left, and projecting right—are *coeval*, in the sense that the introduction and elimination rules for any one of them involves the other two notions.

As one pursues an inferentialist treatment of important logical and mathematical notions in terms of introduction and elimination rules, one finds an interesting variety of 'grades of logicality'. The highest grade is that enjoyed by the most readily understandable logical operators, the connectives (the subject matter of so-called 'baby logic'). These are governed by rules that deal with them one at a time, in isolation from all other operators. The second grade of logicality is that occupied by the quantifiers in free logic, and the logical predicate of existence. These are governed by introduction and elimination rules in whose formulation one is allowed to *use* an earlier logical operator, already independently introduced with *its* own rules. (See [39] for details.) The third grade of logicality is occupied by those notions, like orderly pairing and its associated projections, that are coeval, in the sense that they interanimate, as it were, in their introduction and elimination rules.

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[14]This Schema was formulated in a talk to the Cambridge Moral Sciences Club, and recorded in the minutes for November 1983.

5 Inferentialist accounts of the meanings of logical operators

A crude conventionalist proposal, misappropriating the Wittgensteinian dictum that *meaning is use*, has it that (i) the meaning of a connective can be fully characterized by the rules of inference that govern it, *and* (ii) *any* rules we lay down, as a 'convention', will fix a meaning for the connective that they are supposed to govern. It is (ii) which is in serious error. Arthur Prior wrote his classic paper [25] in an attempt to show that if (ii) were to be conceded, this crude inferentialism would be semantically impotent to distinguish genuine from phoney or freakish logical connectives. There followed the rejoinder by Belnap [1], famously imposing on the deducibility relation certain structural conditions (such as transitivity, or Cut), conditions which (not unreasonably) would need to be in place, as presumed background, before the inferentialist could claim that his chosen rules succeeded in characterizing a logical operator. It is against the background of that early exchange that more recent inferentialists have pursued more fully developed accounts.

5.1 Introduction and elimination of sentence-forming operators

The introduction rule for a sentence-forming logical operator λ (such as a connective or a quantifier) tells one how to *infer to* a compound *conclusion* with λ dominant. In doing so, one *introduces* the dominant occurrence of λ in the conclusion. The elimination rule for λ tells one how to *infer away from* a compound *premise* with λ dominant. In doing so, one *eliminates* the displayed dominant occurrence of λ in the premise (called the *major* premise for the elimination).

5.2 Introduction and elimination of term-forming operators

Now, premises and conclusions of rules of inference for an operator are always sentences. So what happens when the operator in question is not a sentence-forming operator? In that case, it must be a term-forming operator—assuming, with Frege, that *Term* and *Sentence* are the only two basic categories in our categorial grammar. For a term-forming operator α, a salient occurrence will be one that is dominant on one side of a general identity claim of the form '$t = \alpha(\ldots)$'. The operator α may or may not be variable-binding. Variable-binding term-forming operators include

$\#x\Phi(x)$: the number of Φs;

$\{x \mid \Phi(x)\}$: the set of Φs;

$\iota x\Phi(x)$: the Φ.

These were treated in some detail in [37].

5.3 Balance, or equilibrium, between introduction and elimination rules

It is important that one who hears a logically complex sentence sincerely asserted should be able logically to infer *from* it all the information that the asserter ought to have acquired before inferring *to* it. By the same token, it is important that one who undertakes to assert a logically complex sentence should ensure that what any listener would be able logically to infer from it is indeed the case. Viewed this way, the rule (λ-I) states the obligation on the part of any speaker who wishes to assert a compound sentence with λ dominant to his listeners; while the rule (λ-E) states the entitlements enjoyed by any listener who hears such a compound being asserted by the speaker. These obligations and entitlements need to be in balance, or equilibrium.

The question now arises: how best might one explicate this notion of balance? There are three different proposals in the literature:

1. conservative extension of logical fragments by new operators;

2. reduction procedures (and normalizability of proofs);

3. harmony, by reference to strength of conclusion and weakness of major premise.

It is, in the present author's view, interesting and important unfinished business to show that these three proposals are equivalent. Here, they will be explained, and proposal (3) will be improved upon.

6 Explications of balance

6.1 Conservative extension of logical fragments by new operators

Suppose one is contemplating adding a new logical operator λ to one's present language L. On the 'conservative extension' proposal, the new rules governing λ must produce a *conservative extension* of the deducibility relation already established within L by the rules of inference that are already

in use.[15] That is, if $\Delta : \varphi$ is a sequent in L (hence not involving any occurrence of λ), then $\Delta \vdash_{L+\lambda} \varphi$ only if $\Delta \vdash_L \varphi$. The extra logical moves newly permitted within $L + \lambda$ should not afford any new inferential transitions among sentences not involving the new logical operator λ.

One should also require *uniqueness* on the part of the logical operator governed by the new rules.[16] That is, if μ were another new logical operator of the same syntactic type as λ, and were uniformly substituted for λ in all the new rules governing λ, then in the resulting 'double extension' $L+(\lambda+\mu)$, we would have λ and μ *synonymous*—that is, they would be inter-substitutable, *salva veritate*, in all statements of deducibility-in-$L+(\lambda+\mu)$.[17]

6.2   Reduction procedures (and normalizability of proofs)

On the second proposal, the balance between introduction and elimination rules is brought out by the *reduction procedure* for λ.[18] The general shape of such a reduction procedure (for a two-place connective λ) is as follows:

$$\begin{array}{c} \text{Subproofs representing} \\ \underline{\text{the speaker's obligations}}_{(\lambda\text{-I})} \\ \underline{\qquad A\lambda B \qquad}_{(\lambda\text{-E})} \\ \text{Listener's entitlements} \end{array} \quad \mapsto \quad \begin{array}{l} \text{More direct warrant for} \\ \text{the listener's entitlements,} \\ \text{not proceeding } via\ A\lambda B \end{array}$$

The unreduced proof-schema on the left in each case shows $A\lambda B$ standing *both* as the conclusion of (λ-I) *and* as the major premise of (λ-E). In other words, the operator λ is introduced, and then immediately eliminated. Such a sentence-occurrence within a proof is called a *maximal sentence-occurrence*. It represents an unnecessary detour, introducing and then immediately eliminating logical complexity that is not needed for the passage of reasoning to be negotiated.

The reducts on the right respectively show that one cannot thereby obtain anything that one did not already possess. The introduction and elimination rules balance each other. Speakers' obligations and listeners' entitlements are in equilibrium. One may with justification say. with Prawitz, that the elimination rule *exactly inverts* the introduction rule.

One can readily illustrate these ideas in the case of conjunctions $A \wedge B$.

(i) One who hears $A{\wedge}B$ sincerely asserted should be able logically

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[15]This requirement is due to Belnap [1].

[16]This requirement was argued for by Peter Schroeder–Heister and Kosta Došen [11].

[17]This criterion of synonymy of logical operators is due to Timothy Smiley [27].

[18]This proposal is due to Prawitz [22].

to infer from it both A and B: for this is all the information that the asserter ought to have acquired before inferring *to $A \wedge B$.*

(ii) One who undertakes to assert $A \wedge B$ should ensure that both A and B are indeed the case: for A and B are what any listener would be able logically to infer from $A \wedge B$.

This balance between speaker's obligations and listener's entitlements is brought out by the following *reduction procedure* for $\wedge$:

$$
\begin{array}{cc}
\Delta & \Gamma \\
\Pi & \Sigma \\
A & B \\
\hline
\multicolumn{2}{c}{A \wedge B} \\
\multicolumn{2}{c}{A}
\end{array}
\quad \mapsto \quad
\begin{array}{c}
\Delta \\
\Pi \\
A
\end{array}
\qquad\qquad
\begin{array}{cc}
\Delta & \Gamma \\
\Pi & \Sigma \\
A & B \\
\hline
\multicolumn{2}{c}{A \wedge B} \\
\multicolumn{2}{c}{B}
\end{array}
\quad \mapsto \quad
\begin{array}{c}
\Gamma \\
\Sigma \\
B
\end{array}
$$

The unreduced proof-schema on the left in each case shows $A \wedge B$ standing *both* as the conclusion of ($\wedge$-I) *and* as the major premise of ($\wedge$-E). In other words, the operator $\wedge$ is introduced, and then immediately eliminated. The occurrence of $A \wedge B$ is maximal. The reducts on the right respectively show that one cannot thereby obtain anything that one did not already possess.

Note also that each of the reducts on the right of the arrow $\mapsto$ has either Δ or Γ as its set of undischarged assumptions. Whichever one it is, it could well be a *proper subset* of the overall set $\Delta \cup \Gamma$ of undischarged assumptions of the unreduced proof-complex on the left. So with the reduction procedure for $\wedge$ we learn an important lesson: reducing a proof (i.e. getting rid of a maximal sentence occurrence within it) *can in general lead to a logically stronger result.* This is because when Θ is a *proper* subset of Ξ, the argument $\Theta : \varphi$ might be a logically stronger argument than the argument $\Xi : \varphi$. It *will* be a logically stronger argument if one of the sentences in $(\Xi \setminus \Theta)$—that is, the set of members of Ξ that are not members of Θ—does not itself follow logically from Θ. To summarize: by dropping premises of an argument, one can produce a logically stronger argument. And reduction can enable one to drop premises in one's proof of an argument. So reduction is a potentially *epistemically gainful* operation to perform on proofs.

The upshot of this discussion of the logical behavior of the conjunction operator is as follows. The operator has an introduction rule, which states the conditions that must be met in order to be entitled to infer to a conjunctive conclusion. And it has a corresponding elimination rule, which states the conditions under which one is entitled to infer certain propositions from a conjunctive major premise. *Every* logical operator enjoys an Introduction

rule and a corresponding Elimination rule. These rules can be formulated in a two-dimensional graphic way, as was the case with $\wedge$-I and $\wedge$-E above.

One possible drawback with the proposal that one appeal to reduction procedures for an explication of balance (between introduction and elimination rules) is that it does not readily apply to proof systems in which there is a requirement that all proofs be in normal form. In such systems, the pre-images for the reduction procedure will not be proofs; and, often, the reducts will not be, either. (This is because reductions get rid of maximal sentence-occurrences one at a time, not all at once.)

For such systems, in which normality is necessary for proofhood, one can still, however, appeal to a kind of 'normalizability' requirement that is closely enough related to the proposal involving reduction procedures. The idea is really quite simple. Given (*normal*) proofs inviting one to 'apply CUT' n times so as to reap the benefits of transitivity:

$$
\begin{array}{cc}
\Delta_1 & \Delta_n \\
\Pi_1 \ \ldots \ \Pi_n \\
\psi_1 & \psi_n \\
\end{array}
$$
$$
\underbrace{\Delta_0, \psi_1, \ldots, \psi_n}_{\Pi_0}
$$
$$
\varphi
$$

one does not form the obvious (and often abnormal) 'proof' and then try to normalize it. Instead, one simply states that, whenever such normal proofs exist (as just displayed), there will always be a *normal* proof, within the system of rules, of some *subsequent* of the overall sequent $\bigcup_i \Delta_i : \varphi$. If the latter proof were always effectively determinable from the given proofs $\Delta_0, \Delta_1, \ldots, \Delta_n$, then that would be an added bonus. And this, in fact, is what obtains for intuitionistic relevant logic.[19]

6.3 Harmony, by reference to strength of conclusion and weakness of major premise

With this much behind us by way of proof-theoretic background, the rest of this study will focus on problems facing an alternative explication of the notion of *balance* or *matching* between, one the one hand, the inferential conditions to which one is beholden in an introduction rule, and, on the

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[19]For the author's defence against John Burgess's claim (see [7], [8]) that this metalogical result cannot be obtained using *IR* as one's metalogic, see [38].

other hand, the conditions of inferential entitlement that are set out in the corresponding elimination rule. In framing a *Principle of Harmony*, one is seeking to capture, in another way, the aforementioned balance that should obtain—and, in the case of the usual logical operators, does obtain—between introduction and elimination rules.

In [28], such a Principle of Harmony was formulated, and this formulation was subsequently refined, first in [30] and later in [34]. The reader will find below summaries of these earlier formulations of the Principle of Harmony. The purpose here is to revisit those formulations in light of an interesting example furnished by Crispin Wright.[20] Wright's tonkish example concerns only sentential connectives; but the lesson to be learned should generalize also to rules governing quantifiers, and (with the obvious necessary modifications) to rules governing term-forming operators.

7   Earlier formulations of Harmony

7.1   The formulation in *Natural Logic*

The original formulation of the Principle of Harmony ([28], at p. 74) was as follows.

> Introduction and elimination rules for a logical operator λ must be formulated so that a sentence with λ dominant expresses the *strongest* proposition which can be inferred from the stated premises when the conditions for λ-introduction are satisfied; while it expresses the *weakest* proposition possible under the conditions described by λ-elimination.

By 'proposition' one can understand 'logical equivalence class of sentences'. Thus when one speaks of the 'proposition' φ, where φ is a sentence, one means the logical equivalence class to which φ belongs.

The *strongest* proposition with property P is that proposition θ with property P such that any proposition σ with property P is deducible from θ; while the *weakest* proposition with property P is that proposition θ with property P that can be deduced from any proposition σ with property P.

Strictly speaking, one should continue to speak here of logical equivalence classes of sentences, or of propositions, but any occasional laxer formulation involving reference only to sentences is unlikely to cause confusion.

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[20]Personal communication, April 2005.

Let us illustrate the foregoing formulation of the Principle of Harmony by reference, once again, to conjunction. In this case, the principle dictates that

> a sentence $\varphi \wedge \psi$ expresses the *strongest* proposition which can be inferred from the premises φ, ψ; while it expresses the *weakest* proposition from which φ can be inferred and ψ can be inferred.

How might this be established? First, observe that by ($\wedge$-I), the sentence $\varphi \wedge \psi$ can be inferred from the premises φ, ψ. In order to establish that $\varphi \wedge \psi$ is the *strongest* proposition that can be so inferred, suppose that θ can also be so inferred. But then by the proof

$$\frac{\dfrac{\varphi \wedge \psi}{\varphi} \quad \dfrac{\varphi \wedge \psi}{\psi}}{\theta}$$

with its two applications of ($\wedge$-E), we have that $\varphi \wedge \psi$ logically implies θ. Thus $\varphi \wedge \psi$ is the *strongest* proposition that can be inferred from φ, ψ.

Secondly, observe that by ($\wedge$-E), the proposition $\varphi \wedge \psi$ is one from which φ can be inferred and ψ can be inferred. In order to establish that $\varphi \wedge \psi$ is the *weakest* proposition from which φ can be inferred and ψ can be inferred, suppose that θ is a proposition from which φ can be inferred and ψ can be inferred. But then by the proof

$$\frac{\dfrac{\theta}{\varphi} \quad \dfrac{\theta}{\psi}}{\varphi \wedge \psi}$$

with its application of ($\wedge$-I), we have that θ logically implies $\varphi \wedge \psi$. Thus $\varphi \wedge \psi$ is the *weakest* proposition from which φ can be inferred and ψ can be inferred.

Now let us provide a foil for the harmony requirement, to illustrate that is has some teeth. Take Prior's infamous connective 'tonk' (here abbreviated as @) with its rules

(@-I) $\quad \dfrac{\varphi}{\varphi @ \psi}$

(@-E) $\quad \dfrac{\varphi @ \psi}{\psi}$

How would one show that $\varphi @ \psi$ is the *weakest* proposition from which ψ can be inferred? It cannot be done. For, suppose θ is a proposition from which

ψ can be inferred. The task now is to show that θ logically implies $\varphi@\psi$. One's first thought might be to use the introduction rule to show this; but, as inspection quickly reveals, that would require φ, not θ, as a premise.

Of course, one could simply *cheat* by saying that θ logically implies $\varphi@\psi$ because of the proof

$$\dfrac{\dfrac{\theta}{\theta@(\varphi@\psi)}\text{(@-I)}}{\varphi@\psi}\text{(@-E)}$$

And this will quickly set one on the path that is further explored in §7.3.

7.2   The formulation in *Anti-Realism and Logic*

The first main improvement on the foregoing formulation of harmony was offered, in *Anti-Realism and Logic*, in response to an observation by Peter Schroeder–Heister, to the effect that the formulation in *Natural Logic* could not guarantee the uniqueness of the logical operator concerned. The requirements of *strength of conclusion* (called (S)) and *weakness of major premise* (called (W)) remained as the two halves of the harmony condition, now called $h(\lambda i, \lambda e)$. (Note the lowercase 'h'. This was essentially the condition of harmony in *Natural Logic*.) The condition $h(\lambda i, \lambda e)$ was spelled out (in [30], at pp. 96–7) as follows. Note that λ is here assumed, for illustrative purposes, to be a binary connective.[21]

   $h(\lambda i, \lambda e)$ holds just in case:

   (S) $A\lambda B$ is the strongest proposition that it is possible to infer as a conclusion under the conditions described by λi; and

   (W) $A\lambda B$ is the weakest proposition that can feature as the major premise under the conditions described by λe.

It was then noted that

   To prove (S) one appeals to the workings of λe; and to prove (W) one appeals to the workings of λi. The rules are thus required to interanimate to meet the requirement of harmony.

The treatment in [30] then involved laying down a further requirement. This was called 'Harmony', with an uppercase 'H':

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[21]Minor changes of spelling and symbols have been made here.

> Given λE, we determine λI as the *strongest* introduction rule λi such that $h(\lambda i, \lambda E)$; and
> given [the rule] λI, we determine [the rule] λE as the *strongest* elimination rule λe such that $h(\lambda I, \lambda e)$.

By requiring Harmony, the intention was to determine uniquely a simultaneous choice of introduction rule λI and elimination rule λE that would be in mutual harmony (with lowercase 'h'). Harmony ensures a kind of Nash equilibrium between introduction and elimination rules: an ideal solution in the coordinate game of giving and receiving logically complex bundles of information.

7.3   The formulation in *The Taming of The True*

The earlier formulation of harmony (lowercase 'h'), which was common to both *Natural Logic* and *Anti-Realism and Logic*, was strengthened in the following re-statement in [34], at p. 321. In this formulation, the earlier comment about how the rules would interact in the proofs of strength-of-conclusion and weakness-of-major-premise was built into the statement of harmony as an essential feature:

> (S) The conclusion of λ-introduction should be the strongest proposition that can so feature; moreover one need only appeal to λ-elimination to show this; but in ... showing this, one needs to make use of all the forms of λ-elimination that are provided

> (W) The major premiss for λ-elimination should be the weakest proposition that can so feature; moreover one need only appeal to λ-introduction to show this; but in ... showing this, one needs to make use of all the forms of λ-introduction that are provided

Suppose one wished to show that the usual introduction and elimination rules for $\rightarrow$ are in harmony. There would accordingly be two problems to solve:

1. Assume that σ features in the way required of the conclusion $\phi \rightarrow \psi$ of $\rightarrow I$:

$$\frac{\quad}{\phi}(i)$$
$$\vdots$$
$$\frac{\psi}{\sigma}(i)$$

Show, by appeal to $\to E$, that $\phi \to \psi \vdash \sigma$.

2. Assume that σ features in the way required of the major premiss $\phi \to \psi$ of $\to E$:

$$\frac{\phi \quad \sigma}{\psi}$$

Show, by appeal to $\to I$, that $\sigma \vdash \phi \to \psi$.

Problem (1) is solved by the following proof:

$$\frac{\dfrac{\overline{}(1)}{\phi} \quad \phi \to \psi}{\dfrac{\psi}{\sigma}(1)} \to E$$

This proof uses only $\to E$ to show that if σ features like the conclusion $\phi \to \psi$ of $\to I$ then σ can be deduced from $\phi \to \psi$. Thus $\phi \to \psi$ is the strongest proposition that can feature as the conclusion of $\to I$.

Problem (2) is solved by the following proof:

$$\frac{\dfrac{\overline{}(1)}{\phi} \quad \sigma}{\dfrac{\psi}{\phi \to \psi}(1)} \to I$$

This proof uses only $\to I$ to show that if σ features like the major premiss $\phi \to \psi$ of $\to E$ then $\phi \to \psi$ can be deduced from σ. Thus $\phi \to \psi$ is the weakest proposition that can feature as the major premiss of $\to E$.

By way of further example, the following two proof schemata show that the usual introduction and elimination rules for $\vee$ are in harmony:

$$\frac{\phi \vee \psi \quad \dfrac{\dfrac{\overline{}(1)}{\phi}}{\sigma} \quad \dfrac{\dfrac{\overline{}(1)}{\psi}}{\sigma}}{\sigma}(1) \vee E \qquad \frac{\sigma \quad \vee I\dfrac{\dfrac{\overline{}(1)}{\phi}}{\phi \vee \psi} \quad \vee I\dfrac{\dfrac{\overline{}(1)}{\psi}}{\phi \vee \psi}}{\phi \vee \psi}(1)$$

Note how in the second proof schema we need to employ both forms of $\vee I$ in order to construct the proof schema; the reader should consider once again the precise statement of the Principle of Harmony given above.

8 Wright's tonkish example

Wright set out the following problem:[22]

> Let λ be a binary connective associated with rules λI and λE. According to the $AR\&L$ characterisation, these are harmonious just in case
>
>> Condition (i): any binary connective @ for which the pattern of λI is valid may be shown by λE to be such that $A\lambda B \models A@B$ (so $A\lambda B$ is the strongest statement justified by the λI premisses); and
>>
>> Condition (ii): any binary connective @ for which the pattern of λE is valid may be shown by λI to be such that $A@B \models A\lambda B$ (so, in effect, λE is the strongest E-rule justified by the I-rule.)
>
> OK. Let λI be $\vee I$, and λE be $\wedge E$. So, to establish Condition (i), assume $A \models A@B$, and $B \models A@B$. We need to show via λE that $A\lambda B \models A@B$. Assume $A\lambda B$. Then both A and B follow by λE. Either will then suffice for $A@B$, by the assumption. As for Condition (ii), assume $A@B$. By hypothesis, the pattern of λE is valid for @, so we have both A and B. Either will suffice via λI for the proof of $A\lambda B$. So $A@B \models A\lambda B$.
>
> What has gone wrong? Manifestly the λ-rules are disharmonious (in fact they are the rules for tonk, of course.)

9 The revised version of the Principle of Harmony

Perhaps the best way to explain what has gone wrong is to clarify how [34] had already, in effect, put it right. The following emended version of harmony is a minor variation of the theme in [34], and was communicated to Wright in response to this interesting problem. In [34], the statement of (S) (strength-of-conclusion) involved the condition that in establishing strength 'one need only appeal to λ-elimination' and the statement of (W) (weakness-of-major-premise) involved the condition that in establishing weakness 'one

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[22]Direct quote from personal correspondence, with minor typos corrected, extra formatting supplied, and some symbols changed in the interests of uniformity of exposition. The example arose in a graduate seminar at NYU, conducted with Hartry Field in the Spring of 2005, on revision of classical logic.

need only appeal to λ-introduction' . These two conditions will be made more emphatic: respectively, 'one may not make any use of λ-introduction' and 'one may not make any use of λ-elimination'.

In order to distinguish the 'final' version of harmony offered below from its predecessors, let us use (S') and (W') as the respective labels for strength-of-conclusion and weakness-of-major-premise. Wright's apparent counterexample can be rendered inadmissible by laying down the following more emphatic version of the requirement for harmony in [34]. It is in spirit of the original, but now also—one hopes—in appropriately captious letter. As has been the case all along, it is framed by reference to a connective λ. The newly emphasized conditions are in boldface.

> (S') $A\lambda B$ is the strongest conclusion possible under the conditions described by λI. Moreover, in order to show this,
>
> (i) one needs to exploit all the conditions described by λI;
>
> (ii) one needs to make full use λE; but
>
> (iii) **one may not make any use of λI.**
>
> (W') $A\lambda B$ is the weakest major premise possible under the conditions described by λE. Moreover, in order to show this,
>
> (iv) one needs to exploit all the conditions described by λE;
>
> (v) one needs to make full use λI; but
>
> (vi) **one may not make any use of λE.**

Suppose now that we try to follow Wright's foregoing suggestion. That is, suppose we try to stipulate λI as $\vee$-like and λE as $\wedge$-like:

$$(\lambda I) \quad \frac{A}{A\lambda B} \quad \frac{B}{A\lambda B}$$

$$(\lambda E) \quad \frac{A\lambda B}{A} \quad \frac{A\lambda B}{B}$$

It will be shown that this stipulation does not satisfy the joint harmony requirement (S') and (W').

In order to establish (S') we would need to show, *inter alia*, that given the inferences A/C and B/C one could form a proof Π (say) of C from $A\lambda B$, but would (i) need both those inferences to do so, and (ii) need to make full use of λE.

In order to establish (W') we would need to show, *inter alia*, that given the inferences C/A and C/B one could form a proof Σ (say) of $A\lambda B$ from C, but would (i) need both those inferences to do so, and (ii) need to make full use of λI.

Candidates for the sought proof Π (for (S')) might be thought to be

$$\frac{\dfrac{A\lambda B}{A}}{C} \qquad \frac{\dfrac{A\lambda B}{B}}{C} \quad ;$$

but neither of these proofs exploits both the inference A/C and the inference B/C. So these candidate proofs violate requirement (i) in (S') to the effect that one needs to exploit *all* the conditions described by λI. Moreover, each candidate proof uses only 'one half' of λE, not full λE. So these proofs also violate requirement (ii) in (S') to the effect that one must use *all* of λE.

Candidates for the sought proof Σ (for (W')) might be thought to be

$$\frac{\dfrac{C}{A}}{A\lambda B} \qquad \frac{\dfrac{C}{B}}{A\lambda B} \quad ;$$

but neither of these proofs exploits both the inference C/A and the inference C/B. So these candidate proofs violate requirement (i) in (W') to the effect that one needs to exploit *all* the conditions described by λE. Moreover, each candidate proof uses only 'one half' of λI, not full λI. So these proofs also violate requirement (ii) in (W') to the effect that one must use *all* of λI.

Suppose rather that one were to stipulate a perverse choice of 'halves' of the preceding I- and E-rules for λ. Thus suppose that the rules λI and λE were now taken to be, respectively,

$$(\lambda I) \quad \frac{A}{A\lambda B}$$

$$(\lambda E) \quad \frac{A\lambda B}{B}$$

In order to establish (S') we would need to show, *inter alia*, that given the inference A/C one could form a proof Π (say) of C from $A\lambda B$, but without (by (iii)) making any use of λI.

In order to establish (W') we would need to show, *inter alia*, that given the inference C/B one could form a proof Σ (say) of $A\lambda B$ from C, but without (by (vi)) making any use of λE.

It might be thought that a candidate proof for Π would be

$$\dfrac{\dfrac{\dfrac{A\lambda B}{B}{\scriptstyle(\lambda E)}}{B\lambda C}{\scriptstyle(\lambda I)}}{C}{\scriptstyle(\lambda E)}$$

but this violates requirement (iii), since it uses λI.

It might be thought that a candidate proof for Σ would be

$$\dfrac{\dfrac{\dfrac{C}{A\lambda B}{\scriptstyle(\lambda I)}}{A}{\scriptstyle(\lambda E)}}{A\lambda B}{\scriptstyle(\lambda I)}$$

but this violates requirement (vi), since it uses λE.

10   On the requirements of full use of conditions and rules

Salvatore Florio has produced an interesting case which illustrates the need to insist on 'full use' of conditions and rules when establishing the statements (S') and (W') for harmony.

Consider the obviously non-harmonious rules

$$(\lambda I)\ \dfrac{A\quad B}{A\lambda B}\qquad\qquad (\lambda E)\ \dfrac{A\lambda B}{C}$$

where the rule (λE) is like the Absurdity Rule, in that it allows the conclusion C to be *any* sentence one pleases.

First we prove (S'), but without heeding fully the requirements that have been laid down on such a proof.

Assume that D features in the way required of the conclusion $A\lambda B$ of λ-introduction:

$$\dfrac{A\quad B}{D}$$

We are required to show that $A\lambda B \vdash D$. The following proof suffices:

$$\frac{A\lambda B}{D} \; (\lambda E)$$

Next we prove (W'), again without heeding fully the requirements that have been laid down on such a proof.

Assume that D features in the way required of $A\lambda B$ as the major premiss for λ-elimination:

$$\frac{D}{C}$$

We are required to show that $D \vdash A\lambda B$. The following proof suffices:

$$\frac{D}{A\lambda B}$$

These swift proofs, however, as already intimated, are unsatisfactory. The respective reasons are as follows.

The proof of (S') does not avail itself at all of the assumption made at the outset (that D features in the way required of the conclusion $A\lambda B$ of λ-introduction). The technical infraction on the part of this attempted proof of (S') is its violation of condition (i): the proof does not 'exploit all the conditions described by λI'. (In fact, one can see from the very statement of (λE) that $A\lambda B$ is the strongest proposition *tout court*, and not just the strongest proposition that can feature as the conclusion of λ-introduction.)

The proof of (W') is defective also. It does not use the rule (λI), thereby violating condition (v).

What has been called the 'final' formulation of harmony, in terms of (S') and (W'), still sits well with the standard connectives. This is revealed by inspection of the obvious demonstrations of maximum strength-of-conclusion and maximum weakness-of-major-premise, when the dominant operator in question is $\neg$, $\wedge$, $\vee$ or $\rightarrow$. The reader will find that conditions (i), (ii) and (iii) (under (S')) and (iv), (v) and (vi) (under (W')) are all satisfied by those demonstrations. So, in ruling out Wright's counterexample, one does not rule out too much. And, the reader will be happy to learn, the *original* tonk, due to Prior, is ruled out also. All that the final formulation of harmony really does is make absolutely explicit features of the demonstrations in question that usually go unremarked, but which need to be emphasized when confronted with tonkish examples masquerading as genuine connectives.

11 Conclusion to the discussion of harmony

Wright's example brings out nicely why it is that stipulations concerning licit methods of proof (of strength-of-conclusion and of weakness-of-major-premise) must be built into the formulation of harmony. The author is moderately confident that the new formulation above, with its emphases on prohibited resources, will withstand any further attempted counterexamples. This new version of harmony (lowercase 'h') should, of course, still be coupled with the uniqueness condition that was called Harmony (uppercase 'H').

In closing, it should be stressed that the formulation of a Principle of Harmony is not just a technical exercise in proof theory of limited (or no) value to the philosophy of logic and language. On the contrary: armed with a satisfactory account of harmony, the naturalizing anti-realist can venture an interesting account of how logical operators could have found their way into an evolving language. Harmony is a transcendental precondition for the very possibility of logically structured communication. A would-be logical operator that does *not* display harmony (such as, for example, Prior's infamous operator 'tonk') could not possibly be retained within an evolving language after making a first debut. Because the 'deductive reasoning' that it would afford would go so haywire, it would have been rapidly selected against. Only those operators would have survived that were governed by harmoniously matched introduction rules (expressing obligations on the part of assertors) and elimination rules (expressing the entitlements of their listeners). For only they could have usefully enriched the medium by means of which social beings can informatively communicate. Only those operators would have been able to make their way through the selective filter for the growing medium.[23]

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